PHOTON GAS, CONT'D

3/15/23

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CALCNS OF $u(T)_{photon}$, $\bar{N}_{photon}$, P_{photon}

Review of calcn leading to no. of photons in $(\omega, \omega+d\omega)$ vol

0) ΔV ⇐

1) $\frac{\Delta V\, d\underline{k}}{(2\pi)^3}$ = no. of waves in ΔV w/in $\underline{k}$, $\underline{k}+d\underline{k}$

2) $Q_{sys} = \sum e^{-\beta \epsilon_j}$ sys = $\tilde{N}$ replicas N ph cy

3) For N- single N- ~~vtd~~ sys I gas $E_j = n_1\epsilon_1 + n_2\epsilon_2 + \cdots$

4) → $Q_s = \left(\sum_{n_1=0} e^{-\beta n_1 \epsilon_1}\right)\left(e^{-\beta n_2 \epsilon_2}\right) \cdots$

5) $\bar{n}_s$ = mean # I gas ptcls in Q state ϵ_s $= \frac{\sum_{n_s=0} n_s e^{-\beta n_s \epsilon_s}}{\sum_{n_s} e^{-\beta n_s \epsilon_s}}$

$= -\frac{1}{\beta}\frac{\partial}{\partial \epsilon_s}\left(\ln \sum e^{-\beta n_s \epsilon_s}\right)$

I

6) $\sum_{n_s=0} e^{-\beta n_s \epsilon_s} =$

$S_n = 1 + e^{-\beta\epsilon_s} + e^{-2\beta\epsilon_s} + \cdots$

$S_n =$

⇓

∴

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7) $\Rightarrow$ $\bar{n}_{s\,\text{photon}}$ = avg. no. of photons ~~I cont photo~~ in N-ptcl system ~~in Q state ϵ_s~~

form : $\boxed{n_s \cdot 4\pi k^2\, dk\, \Delta V/(2\pi)^3}$ =

8) ∴ (AVG. NO of photons in ϵ_s) · (no. of wave modes in ΔV in range k_s, $k_s + dk_s$) = avg. no. photons in ΔV

10 photons

$\bar{n}_s$ ⇓

pattern recognition

8a) $\boxed{f(|\underline{k}|) \cdot d\underline{k}}$ =

‖

8b) (no. of photons w/ wave no. mag. $|\underline{k}|$ ~~minimum $\Delta\underline{k}$~~ / unit vol. in $|\underline{k}|$ space) · (vol in $\underline{k}$ space)

(2) (each $\underline{k}$ can have one of 2 polarizations)

$$\left[\frac{1}{(e^{\beta\epsilon_s} - 1)(2\pi)^3}\right] \cdot 4\pi k^2\, dk\, \Delta V$$

8c) ∴ $\bar{n}_s$ ‖ ‖ = AVG no. of photons in ΔV w/ $\underline{k} \le |\underline{k}| \le |\underline{k}| + dk$

8d)

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9) ~~$\Psi = \Sigma$~~

$c = \frac{\omega}{k} \Rightarrow k = \frac{\omega}{c} \Rightarrow dk = d\omega / c$

$$\therefore \boxed{\frac{f(\omega)\,d\omega}{\Delta V} = \frac{1}{(2\pi)^3 (e^{\beta\hbar\omega} - 1)} \frac{8\pi\,\omega^2\,d\omega}{c^3}}$$

= avg. no. of photons w/ $\hbar\omega$ between ω and $\omega + d\omega$ / vol

$\omega = \nu\, 2\pi \qquad d\omega = 2\pi\, d\nu$

$$\boxed{\frac{f(\nu)\,d\nu}{\Delta V} = \frac{(8\pi/c^3)\,\nu^2\,d\nu}{(e^{\beta h \nu} - 1)}}$$

= avg no. photons w/ freqs between ν & $\nu + d\nu$ / vol.

II) calcn of $u(T)_{\text{photon}}$

10) $$u = \frac{h\, 8\pi}{c^3} \int_0^\infty \frac{\nu^3\, d\nu}{(e^{\beta h \nu} - 1)}$$

11) $$\bar{u}(\omega, T) = \frac{\hbar\omega\, 8\pi\, \omega^2\, d\omega}{(2\pi c)^3 (e^{\beta\hbar\omega} - 1)}$$

= energy of photons in ΔV having frequencies $(\omega, \omega + d\omega)$

12) $\Rightarrow$ $$u(T) = \int_0^\infty {}'' \; d\omega$$

$$\boxed{u(T)_{\text{photons}} = \frac{\pi^2}{15} \frac{(k_B T)^4}{(c\hbar)^3}}$$

See e.g. Reif Ch. 9

$\bar{n} = \frac{U}{\Delta V} = \Sigma$

13) $\boxed{U_{\substack{\text{photon} \\ \text{black body}}} \propto T^4_{\text{body}}}$ $\Rightarrow$ Stefan Boltzmann Law

III) CALC. OF AVG. NO. OF PHOTONS IN ΔV (of all frequencies $0 < \omega < \infty$)

14) $d\bar{N}(\omega) = \sum_{n_s = 0} \bar{n}_s = \int \frac{1}{(e^{\beta \hbar \omega_s} - 1)}$

$\Delta \bar{N}(\omega) = \frac{8\pi K^2 \, dK \, \Delta V}{(2\pi)^3} \cdot \bar{n}(K)$

(no. of wave modes, = photons w/ K, K + dK) $\cdot$ avg no.

$\Delta \bar{N}(\omega) \doteq \frac{8\pi \, \omega^2 d\omega}{(2\pi c)^3} \frac{1}{e^{\beta \hbar \omega} - 1} \Delta V$

$c = \frac{\omega}{K}$

$\Rightarrow \omega = cK$

$\Rightarrow K^2 = \omega^2 / c^2$

$dK = d\omega / c$

15) $\frac{d\bar{N}(\omega)}{\Delta V} = \frac{8\pi \nu^2 d\nu}{c^3 (e^{\beta h \nu} - 1)}$

$\boxed{\omega = 2\pi\nu}$

16) $\therefore$ $\boxed{\frac{\bar{N}}{\Delta V} = \frac{8\pi}{c^3} \int_0^\infty \frac{\nu^2 d\nu}{(e^{\beta h \nu} - 1)}}$

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17) $dU = TdS - PdV$

$P = \frac{\partial U}{\partial V}\Big|_{S}$.

17) $P = -\frac{\partial A}{\partial V}\Big|_{N,T}$

$A = -kT \ln Q_{sys}$

$\therefore P = kT \frac{\partial \ln Q}{\partial V}\Big|_{N,T}$

IV) CALC'N OF AVE photon pressure exerted on walls of our BOX, volume V (or on subvolume ΔV, having virtual surfaces

18) $Q = \sum e^{-\beta E_j}$

$= \sum_{\{n\}} e^{-\beta(n_1\epsilon_1 + n_2\epsilon_2 + \ldots)}$

$= \left(\sum_{n_1=0} e^{-\beta n_1\epsilon_1}\right)\left(\sum_{n_2=0} e^{-\beta n_2\epsilon_2}\right)\ldots$

$= \left(1 + e^{-\beta\epsilon_1} + e^{-\beta\epsilon_1 2} + e^{-3\beta\epsilon_1} \ldots\right)\ldots$

19) $S_n = A + Af + Af^2 + \ldots$

$A = 1 \quad f = e^{-\beta\epsilon_1}$

$fS_n = Af + Af^2 + Af^3 + \ldots Af^{n+1}$

$S_n - fS_n = A - Af^{n+1}$

$S_n = \frac{1 - f^{n+1}}{1 - f} = \frac{1}{1 - e^{-\beta\epsilon_1}}$

$$\therefore \boxed{Q = Z = (1-e^{-\beta\epsilon_1})^{-1}(1-e^{-\beta\epsilon_2})^{-1} \cdots}$$ 6

$$= \prod_{\epsilon_s=1}(1-e^{-\beta\epsilon_s})^{-1}$$

20) $\epsilon_s = \hbar\omega_s = \hbar c k_s$

$$= \hbar c\left[k_x^2 + k_y^2 + k_z^2\right]_s^{1/2}$$

$$= \hbar c\left[\frac{2\pi}{L}\right](n_x^2 + n_y^2 + n_z^2)^{1/2}$$

$$\Uparrow$$

From $\boxed{\nabla^2 E + k^2 E = 0}$

$$= C_s L^{-1} = C_s V^{-1/3}$$

21) $\therefore \left.\frac{\partial Q}{\partial V}\right| = \frac{\partial}{\partial V}\prod_s (1-\exp[-\beta\epsilon_s])^{-1}$

$$\cancel{= \prod_s (-\beta\epsilon_s}$$

$$\frac{e^{\beta\epsilon_s}}{e^{\beta\epsilon_s}-1} = \frac{1}{\beta}\frac{\partial}{\partial\epsilon_s}\ln(e^{\beta\epsilon_s}-1)$$

$$\frac{\partial}{\partial V}\prod_s \frac{1}{\beta}\frac{\partial}{\partial\epsilon_s}\ln(e^{\beta\epsilon_s}-1)$$

$$= \prod_s \frac{1}{\beta}\frac{\partial}{\partial\epsilon_s}\left[\frac{1}{(e^{\beta\epsilon_s}-1)}\right]\beta\left(\frac{\partial\epsilon_s}{\partial V}\right)$$

$$(2)\ (3)(4)$$

$$= [1](1-e^{-\beta\epsilon_2})\cdots \frac{(1)}{(1)}$$

$$(1)[2](3)(4)$$

$$+ [1-e^{-\beta\epsilon_1}][2] \cdot \frac{(2)}{(2)} (1-e^{-\beta\epsilon_3})\cdots$$

$$+ (1)(2)[3](4)\frac{(3)}{(3)}$$

(7.1) $$\boxed{\frac{\partial Q}{\partial V} = Q\left[\sum_{s=1} \frac{[s]}{(s)}\right]}$$

(7.2) $$(s) = \left(1 - e^{-\beta\epsilon_s}\right)^{-1}$$

(7.3) $$[s] = \frac{\partial \left(1 - e^{-\beta\epsilon_s}\right)^{-1}}{\partial V}$$

$$= (-1)(s)^2 \frac{\partial}{\partial V}\left(-e^{-\beta\epsilon_s}\right) = (-1)(s^2)\frac{\partial}{\partial \epsilon_s}\left(-e^{-\beta\epsilon_s}\right)\frac{\partial \epsilon_s}{\partial V}$$

$$= (-1)(s^2)\beta\epsilon_s \frac{\partial \epsilon_s}{\partial V} \qquad \boxed{(-1)\beta s^2 e^{-\beta\epsilon_s}\frac{\partial \epsilon_s}{\partial V}}$$

$$\therefore \frac{\partial \ln Q}{\partial V} = -\beta \sum_{s=1} s \frac{\partial \epsilon_s}{\partial V}$$

$$= -\beta \sum_s \frac{e^{-\beta\epsilon_s}}{\left(1 - e^{-\beta\epsilon_s}\right)}\left(\frac{\partial \epsilon_s}{\partial V}\right)$$

(7.4) $$\boxed{\frac{\partial \ln Q}{\partial V} = -\beta \sum_s \left(\frac{1}{e^{\beta\epsilon_s} - 1}\right)\frac{\partial \epsilon_s}{\partial V}} = \boxed{-\beta \sum_s \bar{n}_s \frac{\partial \epsilon_s}{\partial V}}$$

(7.5) $$\therefore P = -\left.\frac{\partial A}{\partial V}\right|_{N,T}$$

$$= -\frac{\partial}{\partial V}\left[kT \ln Q\right]\Big|_{N,T}$$

$$= (-kT)\left.\frac{\partial \ln Q}{\partial V}\right|_{N,T}$$

$$= \sum \bar{n}_s \frac{\partial \epsilon_s}{\partial V}$$