(Dervn. Bose-Einstein distbn)

IN-CLASS NOTES 3/22/23

①

I) $\boxed{2 \text{ LIMITS}}$ FOR N-PARTICLE NON-INTERACTING SYSTEMS

("I GAS")

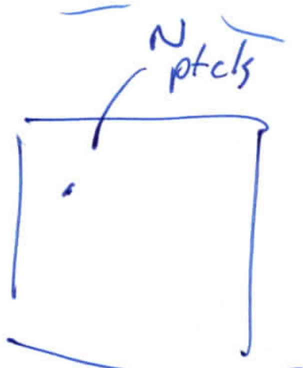
A) LOW TEMP AND/OR PTCL DENSITY HIGH $\Rightarrow$ QUANTUM STATISTICS

i) I GAS OF Fermions:

$$\boxed{\bar{n}_{s_{FD}} = \frac{1}{e^{\beta(\epsilon_s - \mu)} + 1}} \quad (1.1)$$

= FERMI-DIRAC DISTBN

(+) $\bar{n}_s =$



@ t:

$N = n_1 + n_2 + n_3$

$E = n_1 \epsilon_1 + n_2 \epsilon_2 + \dots$

ii) I GAS Bosons:

$$\boxed{\bar{n}_{s_{BE}} = \frac{1}{e^{\beta(\epsilon_s - \mu)} - 1}} \quad (1.2)$$

B) LIMIT 2: $\boxed{T \text{ high}}$ and/or ptcl density $\boxed{\text{LOW}}$

2

Boltzmann / classical Distbn =

$$\boxed{\bar{n}_s = \frac{e^{-\beta \epsilon_s}}{\sum_{s=0} e^{-\beta \epsilon_s}}} \quad (2.1)$$

a) (2.1) $\Longleftarrow$ High T / Low N limit of (1.1) and (1.2)

b) most work-a-day physics problems corresponds to this limit

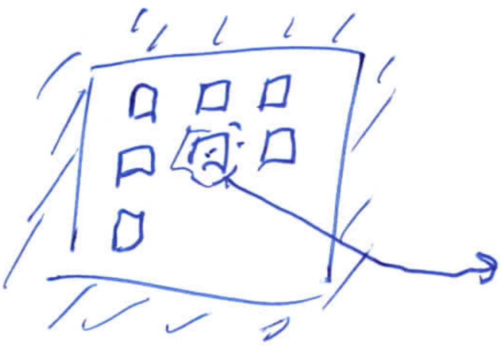
II) Bose-Einstein distbn (3)

o) ~~Using~~ using GRAND-CANONICAL DISTBN FN

A)

(3.1)
$$p(E_j, N) = \frac{e^{-\beta(E_j)} e^{\beta\mu N}}{\sum_{j,N} e^{-\beta E_j} e^{\beta\mu N}}$$

⊕ LHS (3.1) ⟺ Joint prob. of observing a given system in ~~quantum~~ system QUANTUM ENERGY $\boxed{E_j}$ and system ptcl no. = $\boxed{N}$



⊕ (3.1) ⇒ NONINTERACTING AND INTERACTING

B) FOR N-ptcl non-interacting identical system

time t:

$$N = n_1 + n_2 + n_3 + \dots$$

(3.2)
$$E_j = n_1 \epsilon_1 + n_2 \epsilon_2 + n_3 \epsilon_3 + \dots$$

C)

D) (3.2) into (3.1);

1, 5, 5003

4.1 $$P(n_1, n_2, n_3 \ldots) = \frac{\left[e^{-B(n_1\epsilon_1 + n_2\epsilon_2 + \ldots)} \cdot e^{uB(n_1+n_2+n_3+\ldots)} \right]}{\sum_{\{n\}} e^{-B(n_1\epsilon_1+\ldots)} e^{uB(n_1+n_2+\ldots)}}$$

$\|$

joint prob. of observing $\boxed{n_1}$ ptcls ~~systems~~ in single Q state $\boxed{\epsilon_1}$, n_2 in s.Q. state $\boxed{\epsilon_2}$

etc.

E) since ptcls are non-interacting

(4.1) $\Rightarrow$

4.2 $$P(n_1, n_2 \ldots) = \left(\frac{e^{-Bn_1(\epsilon_1 - u)}}{\sum_{n_1=0} e^{-Bn_1(\epsilon_1 - u)}} \right) \cdot \left(\frac{e^{-Bn_2(\epsilon_2 - u)}}{\sum_{n_2=0} e^{-Bn_2(\epsilon_2 - u)}} \right) \ldots$$

F) (4.2) $\Rightarrow$ $\begin{pmatrix}\text{JOINT} \\ \text{prob.}\end{pmatrix} = \begin{pmatrix}\text{prob of } n_1 \text{ ptcls} \\ \text{(RV.) in } \epsilon_1\end{pmatrix} \begin{pmatrix}\text{prob of obsing} \\ n_2 \text{ in state } \epsilon_2\end{pmatrix}$

(5)

G) or $\boxed{p(n_1, n_2, n_3 \ldots) = p_{\epsilon_1}(n_1) \cdot p_{\epsilon_2}(n_2) \cdot p_{\epsilon_3}(n_3)}$ (5.1)

$P_{\epsilon_s}(n_s) =$ prob. of obtaining (given no.) n_s ptcls in a state ϵ_s

H) $$\boxed{\bar{n}_s = \frac{\sum_{n_s=0} e^{-\beta n_s(\epsilon_s - \mu)}\, n_s}{\sum_{n_s=0} e^{-\beta n_s(\epsilon_s - \mu)}}}$$ (5.2)

(V.1) for $\bar{n}_s$

I-gas of Fermions or Bosons

~~III~~ III) Boson I-GAS system

A) $$\boxed{\bar{n}_s = \frac{\sum_{n_s=0} e^{-\beta n_s(\epsilon_s)}\, \lambda^{n_s}\, n_s}{\sum_{n_s=0} e^{-\beta n_s \epsilon_s}\, \lambda^{n_s}}}$$ (V.2) . (5.3)

where $\boxed{\lambda \equiv e^{\mu\beta}}$

B) (5.8) can be rewritten as:

$$(6.1)\quad \boxed{\bar{n}_s = \lambda \frac{\partial}{\partial \lambda}\Big[\ln \overbrace{\sum_{n_s} \left(e^{-\beta \epsilon_s n_s} \lambda^{n_s}\right)}^{F}\Big]}$$

proof (6.1):

i) $$RHS = \lambda \frac{\sum_{n_s} e^{-\beta \epsilon_s n_s} n_s \lambda^{n_s - 1}}{\sum_{n_s} e^{-\beta \epsilon_s n_s} \lambda^{n_s}}$$

$$= \bar{n}_s \qquad \text{proved}$$

C) i) $$F = \sum_{n_s=0} e^{-\beta n_s(\epsilon_s - \mu)} = 1 + (1)e^{-\beta(\epsilon_s-\mu)(1)} + (1)e^{-\beta(\epsilon_s-\mu)(2)} + \ldots$$

Reif Appendix $A = 1,\ f = e^{-\beta(\epsilon_s - \mu)}$

ii) $$F = S_n = A + Af + Af^2 + \ldots + Af^n$$

$-$

iii) $$f S_n = fA + Af^2 + Af^3 + \ldots + Af^{n+1}$$

iv) $$S_n - f S_n = A - Af^{n+1}$$

(7)

v) $S_{n \to \infty} = \dfrac{\cancel{A}(1 - \cancel{t^{n+1}})}{1 - t}$

vi) $\boxed{F = \dfrac{1}{1 - e^{-\beta(\epsilon_s - \mu)}}}$ (7.1)

D) ∴ (7.1) in (6.1)

$$\bar{n}_s = \lambda \frac{\partial}{\partial \lambda}\left(\ln\left[1 - e^{-\beta\epsilon_s}\lambda\right]\right)$$

$$= (-\lambda)\frac{(-1)e^{-\beta\epsilon_s}(1)}{[1 - e^{-\beta\epsilon_s}\lambda]}$$

$$= \frac{e^{-\beta(\epsilon_s - \mu)}}{1 - e^{-\beta(\epsilon_s - \mu)}}$$

$$\boxed{\bar{n}_{s\,\text{B.E.}} = \frac{1}{e^{\beta(\epsilon_s - \mu)} - 1}}$$

* Bose-Einstein distbn
for N-non interacting
Bosons, Any temp
Any N

PREP notes 3/22/23

$$\bar{n}_s = \frac{1}{e^{B(\epsilon_s - \mu)} \pm 1}$$

$$e^{B(\epsilon_s-\mu)} \gg 1 \Rightarrow \frac{e^{B\epsilon_s}}{e^{B\mu}} \gg 1 \quad (1)$$

$$\Downarrow$$

$$e^{B\epsilon_s} \gg e^{B\mu}$$

$$\Downarrow$$

$$\bar{n}_s =$$

$$\bar{n}_s \ll 1 \Rightarrow \nearrow \quad \bar{n}_s \sim e^{-B\epsilon_s}$$

$$\bar{n}_s \propto e^{-B\epsilon_s}$$

$$\bar{n}_s = \frac{\sum_{n_s} n_s e^{-n_s B(\epsilon_s - \mu)}}{\sum_{n_s=0} e^{-n_s B(\epsilon_s-\mu)}} = \lambda \frac{\partial}{\partial \lambda} \ln \sum_{n_s} \left(e^{-n_s B\epsilon_s} \lambda^{n_s}\right)$$

$$= \frac{\sum_{n_s} n_s e^{-n_s B\epsilon_s} \lambda^{n_s}}{\sum_{n_s} e^{-n_s B\epsilon_s} \lambda^{n_s}} \checkmark$$

$$\sum_{n_s=0} e^{-n_s B(\epsilon_s-\mu)} = 1 + e^{-B(\,)(1)} + e^{-B(\,)2} + \dots$$

$$S_n = A + Af + Af^2 + Af^3$$

$$fS_n = Af + Af^2 + \qquad -$$

$$\cancel{fS_n - S_n}$$

$$S_n(1-f) = A - Af^{n+1}$$

$$S_{n\to\infty} = \frac{1}{1-f} = \frac{1}{1-e^{-B(\epsilon_s-\mu)}} = \frac{1}{1-e^{-B\epsilon_s}\lambda}$$

$$= \frac{e^{B(\epsilon_s-\mu)}}{e^{B(\epsilon_s-\mu)}-1}$$

$$\Downarrow$$

$$\bar{n}_s = -\lambda \frac{\partial}{\partial\lambda} \ln(1-e^{-B\epsilon_s}\lambda)$$

$$\therefore \quad \bar{n}_s = \lambda \frac{\partial}{\partial \lambda} \ln \frac{[e^{B\epsilon_s}\lambda^{-1}]}{\boxed{e^{B\epsilon_s}\lambda^{-1} - 1}} =$$

$$= -\lambda \frac{(-e^{-B\epsilon_s})}{1-e^{-B\epsilon_s}\lambda}$$

$$= \frac{1}{e^{B(\epsilon_s-\mu)}-1}$$

(2)

$$= \sum_{j,N} e^{-\beta E_j + \mu N}$$

$$\boxed{\tilde{\Sigma} = \sum_{n_s} e^{-\beta(n_s)(\epsilon_s - \mu)}} \quad (1)$$

$$E_j = n_1 \epsilon_1 + n_2 \epsilon_2 + \ldots$$

$$N = n_1 + n_2$$

$$= e^{-\beta n_1^{0}(\epsilon_1 - \mu)} + e^{-\beta n_1^{1}(\epsilon_1 - \mu)}$$

$$+ e^{-\beta n_2^{0}(\epsilon_2 - \mu)} + e^{-\beta n_2^{1}(\epsilon_2 - \mu)}$$

$$= (1 + e^{-\beta \epsilon_1}\lambda) \times (1 + e^{-\beta \epsilon_2}\lambda) \times$$

$$\ldots (1 + e^{-\beta \epsilon_3}\lambda) \times (\quad)$$

As before $\bar{n}_s = \sum$

$\tilde{\Sigma}$ $$\bar{n}_s = \frac{\sum_{n_s} n_s e^{-n_s \beta(\epsilon_s - \mu)}}{\sum e^{-n_s \beta(\epsilon_s - \mu)}}$$

meaning – only one part can be in each Q state ϵ_s

$$\tilde{\Sigma} = \prod_s \xi_s \quad \xi_s = 1 + e^{-\epsilon_s \beta}\lambda$$

(2) But for fermions — partition fn for a state ϵ_1

$$\tilde{\Sigma} = q_1 \cdot q_2 \cdot q_3 \ldots$$ ← this explains pauli exclusion principle

$$= (e^{-\beta(\epsilon_1 - \mu) n_1^{0}} + e^{-\beta(\epsilon_1 - \mu) n_1^{1}})(1 - e^{-\beta(\epsilon_2 - \mu)})$$

$$\tilde{\Sigma} = \prod_s (1 - e^{-\beta \epsilon_s}\lambda)$$

$$P_{Fr.}(n_s) = \frac{e^{-\beta n_s(\epsilon_s-\mu)}}{\sum_{n_s=0}^{1} e^{-\beta n_s(\epsilon_s-\mu)}} = \frac{e^{-\beta n_s(\epsilon_s-\mu)}}{1+e^{-\beta n_s(\epsilon_s-\mu)}}$$

$$P_s(0) = \frac{1}{1+e^{-\beta(\epsilon_s-\mu)}} =$$ prob of obs'ing $\boxed{0}$ Fermions in $\boxed{\epsilon_s}$

$$P_s(1) = \frac{e^{-\beta(\epsilon_s-\mu)}}{1+e^{-\beta(\epsilon_s-\mu)}} =$$ prob of obs'ing $\boxed{1}$ Fermion in $\boxed{\epsilon_s}$

$$\bar{n}_s = P(0)\cdot(n_s=0) + P(1)(n_s=1)$$

$$= \frac{e^{-\beta(\epsilon_s-\mu)}}{1+e^{-\beta(\epsilon_s-\mu)}} = \frac{1}{(e^{\beta(\epsilon_s-\mu)}+1)}$$

$\Downarrow$

Fermi Dirac Distbn

$\parallel$

avg no. Fermions in state ϵ_s

$\overline{n}_s$ ~~low~~ $\Rightarrow$

pg. 63 Hill
Important (classical / Boltzmann)
Limiting Forms of
$\boxed{B-E}$ and $\boxed{F-D}$ statistics
occurs when no of ~~ptcls~~
~~occurs~~ indep. ptcls, $\boxed{N}$, is
$h\nu_{max} \sim 10 k_B T$ or $\epsilon_{s_{max}} \sim 10 k_B T$ much smaller than No of
Q states avail, $\boxed{N_{QS}}$ to N-ptcl.
(~~[illegible]~~ INDEP) system:
$\boxed{N_{ptcl} \ll N_{QS}}$

$\Updownarrow$
where
$\boxed{\epsilon_0 \leq \epsilon_s \lesssim 10 k_B T}$
$\Updownarrow$
$\boxed{T \gtrsim \epsilon_s / 10 k_B}$
$\Updownarrow$
$k_B T \gtrsim \frac{h\nu}{10}$

in this case, ~~for [illegible]~~
Terms $n_1, n_2, n_3 \ldots$
in
$E_{sys} = n_1 \epsilon_1 + n_2 \epsilon_2 + n_3 \epsilon_3 +$
will be mostly 0 w/
a few non-zero terms

(5)

I) $\boxed{N_{QS}} \equiv$ No. of AVAIL Q STATES $\Leftrightarrow\ \approx$ $0 \leq \epsilon_0 <$ ptcl. QSTATE energies, ϵ_s $\lesssim 10 k_B T$

"low density limit" = Limit & requires

A) $\boxed{\text{IF } N_{ptcl} << N_{QS}} \Leftarrow$ T high enough

Then most terms $\boxed{n_s}$ in:

$E_{sys} = n_1 \epsilon_1 + n_2 \epsilon_2 + n_3 \epsilon_3 + \ldots$

ARE $\boxed{0}$

B) $\therefore$ $\boxed{\bar{n}_s \to 0} \Rightarrow$

i) $\boxed{\text{B.E CASE}}$: $\bar{n}_s = \dfrac{1}{e^{\beta(\epsilon_s - \mu)} - 1}$ (5.1)

ii) For $\boxed{\bar{n}_s \to 0}$, $e^{\beta(\epsilon_s - \mu)} >> 1$

$$\bar{n}_s = e^{-\beta \epsilon_s} e^{\beta \mu}$$

$$\bar{N} = \sum \bar{n}_s = e^{\beta\mu} \sum e^{-\beta\epsilon_s}$$

$$e^{\beta\mu} = \bar{N} / \sum e^{-\beta\epsilon_s}$$

$$\therefore \boxed{\frac{\bar{n}_s}{\bar{N}} = \frac{e^{-\beta\epsilon_s}}{\sum e^{-\beta\epsilon_s}}}$$ $\boxed{\text{HIGH T Limit}}$

C): For HIGH temp AGAIN $\boxed{N_{QS} >> N}$

so that again n_s mostly 0 =