(In-class notes 3/24/23) ①

I] $\bar{n}_{s}$ v. I GAS (≡ noninteracting ptcls) of Fermions

$$\boxed{\bar{n}_{s_{FD}} = \frac{1}{e^{\beta(\epsilon_s - \mu)} + 1}}$$

Fermi Dirac distbn AND proof that $N_Q \gg N_{ACTUAL}$ criterion is consistent w/ classical statistical mech

$\bar{n}_s$ = AVG no. Fermions in single ptcl quantum energy state $\epsilon_s$

⊕ We get $\epsilon_s$ by solving S.E. for a single ptcl.

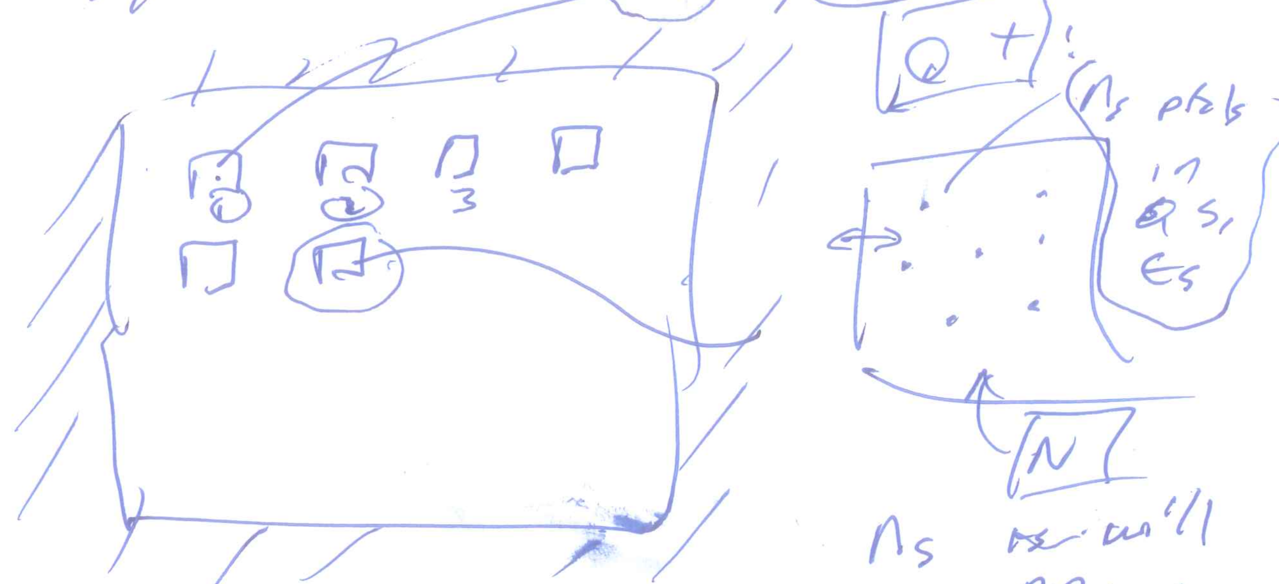
GRAND CANON. ENS.

A.
$$\boxed{\hat{P}(E_j, N) = \frac{\cancel{e^{-\beta(E_j)}} \; e^{-\beta(E_j)} e^{\beta\mu N}}{\sum_{j,N} e^{-\beta(E_j)} e^{\beta\mu N}}} \quad (1.1)$$

⊕ $E_j$ = Quantum eigen-energy for $N$ ptcls in Vol $V$

⊕ $E_j$ ⇒ S.E. for a system of $N$ NONINTERACTING OR INT'ING

N = no. of ptcls in each (2) system




$n_s$ will vary about $\boxed{\bar{n}_s}$

(+) IMP PT:

$P(E_j, N)$ = joint prob. of obs'ing any sys. (w/in GCE) w/ $\bar{N}$ ptcls and $\boxed{E_j}$

B) For NON-INTERACTING (E) FORCES between ptcls (TINY) systems

single ptcl q. energy

(+) (2.1) $E_j = n_1\epsilon_1 + n_2\epsilon_2 + n_3\epsilon_3 + \ldots$

$N = n_1 + n_2 + n_3 + \ldots$

c) ∴ use (2.1) in (2.1) ⇒

$$P(\epsilon_i; N) \Rightarrow P(n_1, n_2, n_3, n_4 \ldots) = P_{\epsilon_1}(n_1) \cdot P_{\epsilon_2}(\epsilon_2) \cdot P_{\epsilon_3}(n_3) \ldots \quad (3.1)$$

R.V. R.V. RV's

where

$$(3.2) \quad P_{\epsilon_s}(n_s) = \frac{e^{-\beta n_s(\epsilon_s - u)}}{\sum_{n_s} e^{-\beta n_s(\epsilon_s - u)}}$$

$P_{\epsilon_s}(n_s)$ = prob of obs'ing $n_s$ ptcls {out of N} at time t in state $\epsilon_s$

D) NOW APPLY (3.1) & (3.2) TO N non-inting Fermions

A) PAULI exclusion principle (4)

$n_s$ either $= \underline{0}$ or $\underline{1}$

B) $$P_{\epsilon_s}(n_s=0) = \frac{1}{\sum_{n_s=0}^{1} e^{-\beta n_s(\epsilon_s-\mu)}}$$

SKIP FOR NOW

a) $$\boxed{P_{\epsilon_s}(n_s=0) = \frac{1}{1+e^{-\beta(\epsilon_s-\mu)}}}$$

b) $$\boxed{P_{\epsilon_s}(n_s=1) = \frac{e^{-\beta(\epsilon_s-\mu)(1)}}{1+e^{-\beta(\epsilon_s-\mu)(1)}}}$$

$\therefore$ c) $$\boxed{\bar{n}_s = \frac{\sum_{n_s=0}^{1} n_s e^{-\beta(\epsilon_s-\mu)n_s}}{\sum_{n_s=0}^{1} e^{-\beta(\epsilon_s-\mu)n_s}}} \quad (4.1)$$

(+) N prtcls are non-interacting

d) or $$\bar{n}_s = \frac{0 \cdot e^{(\,)(0)} + (1)e^{(\,)(1)}}{1 + e^{-\beta(\epsilon_s-\mu)(1)}}$$

$$= \frac{e^{-\beta(\epsilon_s-\mu)}}{1+e^{-\beta(\epsilon_s-\mu)}}$$

∴ $$\boxed{\bar{n}_{s\,\text{Fermions}} = \frac{1}{e^{\beta(\epsilon_s-\mu)}+1}}$$ (5)

- Fermi Dirac Distbn
- Applies to nonintera syst. of $N$ ptls
- Applies to ANY Temp $T$ and and ptcl density $n \equiv N/V$

strictly

III) Hill, Reif and every source:

Hill states that

(5.1)
$$\boxed{\begin{aligned} \bar{n}_{s_{FD}} &\rightarrow \bar{n}_{s\,\text{classical Boltzmann}} \\ \bar{n}_{s_{\text{Boson}}} &\rightarrow \bar{n}_s \quad '' \end{aligned}}$$

where
$$\boxed{\bar{n}_{s\,\text{Boltzman}} = \frac{e^{-\beta\epsilon_s}}{\sum_{\epsilon_s=1} e^{-\beta\epsilon_s}}}$$ ✓

HILL → CLASSICAL limit applies

when $N_{AVAIL}$ QUANTUM states $\gg N$

gas N nonint. ptcl.

+ HILL: MOST STATISTICAL PHYSICS problems (NOT LOW TEMP or super high density) are classical

IV) Quant part. in Box (6)

A) S.E. $\boxed{\hat{H}|\psi\rangle = \epsilon|\psi\rangle}$

B) $\boxed{H = \frac{p \cdot p}{2m}} \Rightarrow$

C) $\Rightarrow \hat{H} \Rightarrow \vec{p} = i\hbar\nabla$

$\Rightarrow \frac{p^2}{2m} \Rightarrow \hbar^2\nabla^2$

D) ∴ A) $\Rightarrow \boxed{-\hbar^2\nabla^2\psi_{p(z)} = \epsilon\psi}$

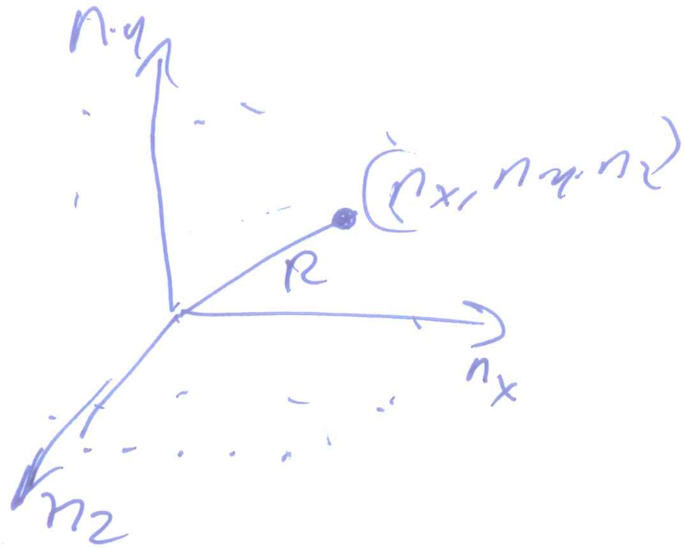
E) $\boxed{\begin{array}{l}\text{B.C.}\\ \psi(\text{walls}) = 0\end{array}}$

F) $\Rightarrow$ ~~Sdn~~ ~~not~~ solve A)

sep n vbls

$\Rightarrow \epsilon$

$L = V^{1/3}$

G) 

OBJ. ⇒ geom arg to est $N_{avail}$ QS single ptcl

$\boxed{N_{QS\ avail}}$

QM
$$\boxed{\epsilon = \frac{h^2}{8mV^{2/3}}\left(n_x^2 + n_y^2 + n_z^2\right)}$$

$n_x = 1, 2, 3,$
$n_y$
$n_z$

H
$$\boxed{R = \sqrt{n_x^2 + n_y^2 + n_z^2} = \left(\frac{\epsilon\, 8mV^{2/3}}{h^2}\right)^{1/2}}$$

$\epsilon \to k_BT$

I) For any T ~~Fixed~~

No of ptcls in states

$\boxed{\epsilon_{s\,max} \gtrsim k_BT}$ are small

⇓

J) Equipartition Thm

⇒ each translation d.o.f. energy = $k_BT$

(8)

$$\boxed{N_{AV}^{QS} \sim \frac{4}{3}\frac{\pi R^3}{8} \sim \frac{4}{3}\pi/}$$

use

(5.1)

when

$$\left(\frac{V}{N}\right)^{1/3} \gg x_{S.E.} \Leftrightarrow \lambda_{DB} = \frac{h}{\sqrt{mk_BT}}$$

$N_{AV/QS} \gg N$

$\Updownarrow$

~~when~~

$N_{AV} \gg N$

when (5.1) holds

$\Updownarrow$

$$\left(\frac{N}{V}\right) \ll 1$$

or

$T \gg 1$

0) system w/in grand canonical ens. having $N$ ptcls and $E_j$

3/24/23 ① PREP WORDS

FOR NON-INTERACTING PTCLS

1) $E_j = n_1\epsilon_1 + n_2\epsilon_2 + n_3\epsilon_3 +$ (1.1)

$N = n_1 + n_2 + n_3$ (1.2)

2) $$P(n_1, n_2, n_3, \ldots) = \frac{e^{-\beta E_j + \mu\beta N}}{\sum_{j,N} e^{-\beta(E_j - \mu N)}}$$ (1.3)

joint prob of obsing a specific set of $n_1, n_2, \ldots$

3) use (1.1), (1.2) in (1.3) $\Rightarrow$

(1.4) $P(n_1, n_2 \ldots) = P_{\epsilon_1}(n_1) \cdot P_{\epsilon_2}(n_2) \cdot P_{\epsilon_3}(n_3) \cdots$

where $P_{\epsilon_s}(n_s)$ = prob. of obsing $n_s$ ptcls in single ptcl E. state $\epsilon_s$

$$P_{\epsilon_s}(n_s) = \frac{e^{-\beta(\epsilon_s - \mu)n_s}}{\sum_{n_s=0} e^{-\beta(n_s)(\epsilon_s - \mu)}}$$ (1.5)

4) FOR 1 GAS of (N) Fermions.

any given ~~single ptcl~~ quantum single ptcl state is either unoccupied $\boxed{n_s = 0}$ or has (at most) 1 ptcl $\boxed{n_s = 1}$ $\Leftarrow$ PAULI EXCLUSION principle

5) ∴ from (1.5)

$$P_{\epsilon_s}(n_s=0) = \frac{1}{\sum_{n_s=0}^{1} e^{-\beta n_s(\epsilon_s-\mu)}} \qquad (2.0)$$

or

$$\boxed{P_{\epsilon_s}(n_s=0) = \frac{1}{1+e^{-\beta(\epsilon_s-\mu)}}} \qquad (2.1)$$

$$\boxed{P_{\epsilon_s}(n_s=1) = \frac{e^{-\beta(\epsilon_s-\mu)}}{1+e^{-\beta(\epsilon_s-\mu)}}} \qquad (2.2)$$

6) ∴ $\bar{n}_{s_{FD}} = \sum_{n_s=0}^{1} n_s P_{\epsilon_s}(n_s)$

$$= 0 \cdot P_{\epsilon_s}(n_s=0) + (1)\, P_{\epsilon_s}(n_s=1)$$

∴

$$\boxed{\bar{n}_{s_{FD}} = \frac{e^{-\beta(\epsilon_s-\mu)}}{1+e^{-\beta(\epsilon_s-\mu)}}}$$

or

$$\boxed{\bar{n}_{s_{FD}} = \frac{1}{e^{\beta(\epsilon_s-\mu)}+1}} \qquad (2.3)$$

(+) Fermi-Dirac distribution for system of non-interacting fermions

(+) HOLDS FOR <u>ANY</u> T and any N as long as ptcls are non-interacting

$N_{QS} >> N$

classical stat thermo

AVAIL

$n_x^2 + n_y^2 + n_z^2 =$

Boltzmann

$n_i = \bar{n}_S$

translation

1) single ptcl in box

~~heat~~ no potential

$\frac{p^2}{2m} |\psi\rangle = \epsilon |\psi\rangle$

2) $$\boxed{\epsilon_{V, n_x n_y n_z} = \frac{h^2}{8 m L^2 \to V^{2/3}} \left( n_x^2 + n_y^2 + n_z^2 \right)}$$ SE

$$\Rightarrow \quad R = \left[ \frac{\epsilon\, 8 m V^{2/3}}{h^2} \right]^{1/2} = \sqrt{n_x^2 + n_y^2 + n_z^2}$$

3) $\Rightarrow$ no. Q states ~~for~~ avail for given $\epsilon \equiv \phi$

Starts

Hill's very leading to $N_{QS\,AVAIL} >> N$ where

$\Downarrow$

$\bar{n}_{SFD} \to \bar{n}_{Boltzmann}$

$\bar{n}_{SBE} \to$ "

$$\phi(\epsilon) \cong 4\pi R^2 dR$$

$$= 4\pi [\;\;]^{1/2} [\;\;]^{1/2} \left( \frac{d\epsilon}{\epsilon} \right)$$

$$= \frac{4\pi}{2} [\;\;]^{3/2} \left( \frac{d\epsilon}{\epsilon} \right)$$

$$\phi = \frac{4}{3} \frac{\pi R^3}{8}$$

positive octant $n_x n_y n_z$-space

$$\boxed{\phi = \frac{\pi}{6} \left( \frac{\epsilon\, 8m}{h^2} \right)^{3/2} V}$$

4) FOR $\bar{n}_{SFD}$ or $\bar{n}_{SBE} << 1$

$\phi$ - f or $\epsilon \sim k_B T$ — Q STATES, weakly

$\epsilon \geq O(k_B T)$ ~~not~~ very

A2

p i p l ... | —

$$\Phi(\varepsilon \sim k_B T) \overset{\text{MUST}}{>>} N_{ptcl}$$

or $\frac{\pi}{6}\left(\frac{k_B T\, 8M}{h^2}\right)^{3/2} V >> N$

or $\left(c_1 \frac{m k_B T}{h^2}\right)^{3/2} >> \frac{N}{V}$

or $\boxed{\frac{h}{\sqrt{c_1 m k_B T}} << \left(\frac{V}{N}\right)^{1/3}}$

or $\boxed{x_{s\sigma} << \ell_{ptcl\ \text{spacing}}}$