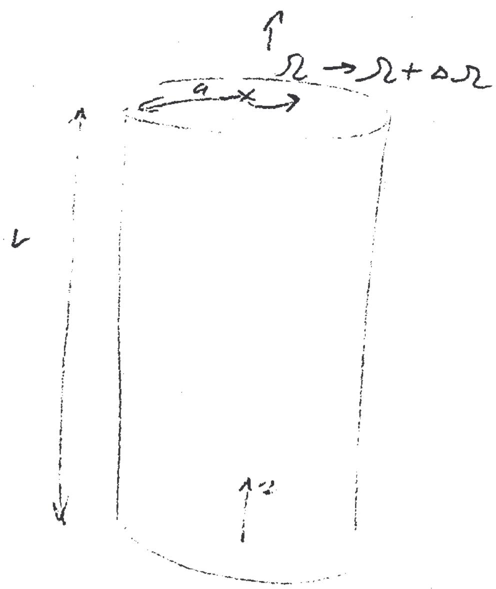
10/16/97 (1)

SPIN-UP OF FLUID IN A ROTATING CYLINDER

QUESTION — HOW LONG DOES IT TAKE FOR FLUID IN ROTATING CYLINDER TO ASSUME A NEW ROTATION RATE

- ILLUSTRATES PHYSICAL REASONING TO OBTAIN PROBLEM SCALES
- ILLUSTRATES HOW BOUNDARY LAYERS CAN DETERMINE CHARACTER OF ENTIRE FLOW (BOUNDARY LAYER + NON-B.L.)
- ILLUSTRATES ASYMPTOTIC METHOD FOR SOLVING FLOW PROBLEMS



FLUID-FILLED CYLINDER CHANGES ROTATION RATE FROM Ω TO $\Omega + \Delta\Omega$.

ASSUME

$$\boxed{\Omega^{-1} \ll \frac{a^2}{\nu}} \quad (1)$$

(i.e., ROTATION PERIOD, $1/\Omega$, MUCH SHORTER THAN time req'd FOR vorticity (created at vertical walls) to diffuse across cylinder)

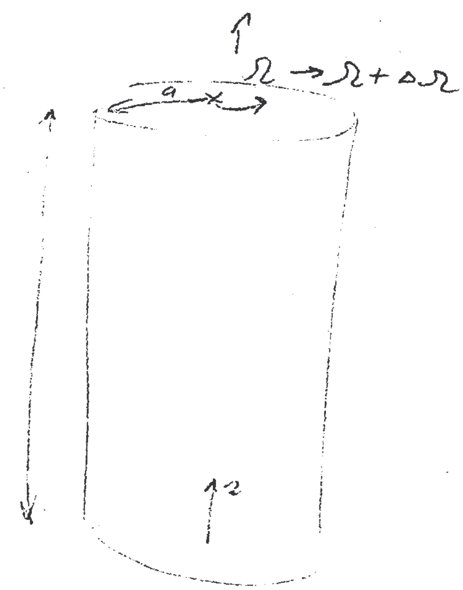
NOTE: PRIOR TO $t=0$ (when $\Omega \to \Omega + \Delta\Omega$), THE FLUID IS IN A STATE OF SOLID BODY ROTATION

10/16/99 ①

SPIN-UP OF FLUID IN A ROTATING CYLINDER

QUESTION — HOW LONG DOES IT TAKE FOR FLUID IN ROTATING CYLINDER TO ASSUME A NEW ROTATION RATE

- ILLUSTRATES PHYSICAL REASONING TO OBTAIN PROBLEM SCALES
- ILLUSTRATES HOW BOUNDARY LAYERS CAN DETERMINE CHARACTER OF ENTIRE FLOW (BOUNDARY LAYER ⊕ NON-B.L.)
- ILLUSTRATES ASYMPTOTIC METHOD FOR SOLVING FLOW PROBLEMS



FLUID-FILLED CYLINDER CHANGES ROTATION RATE FROM Ω TO $\Omega + \Delta\Omega$.

ASSUME

$$\boxed{\Omega^{-1} \ll \frac{a^2}{\nu}} \quad (1)$$

(i.e., ROTATION PERIOD, $1/\Omega$, MUCH SHORTER THAN time req'd for vorticity (created at vertical walls) to diffuse across cylinder)

NOTE: PRIOR TO $t=0$ (when $\Omega \to \Omega + \Delta\Omega$), THE FLUID IS IN A STATE OF SOLID BODY ROTATION

10/16/98

PROCEDURE

1) PHYSICAL PICTURE OF FLOW ADJUSTMENT PROCESS → TIME, LENGTH, Velocity scales

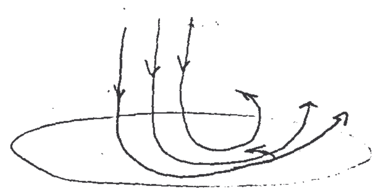
2) BASED ON TIME, LENGTH, VELOCITY SCALES OBT'D IN 1), NON-DIMENSIONALIZE THE NAV. STOKES EQNS. — 2 APPROACHES

3) ASSUME AN ASYMPTOTIC SOLN IN A SMALL PARAMETER ϵ = ECKMAN NUMBER — 2 APPROACHES

4) SOLVE LEADING ORDER FLOW IN BOUNDARY LAYERS AND INVISCID CORE

PHYSICAL PICTURE — VELOCITY, LENGTH, TIME SCALES:

WHEN A FLUID-FILLED CYLINDER STARTS TO SPIN, FLUID NEAR THE ENDS IS FLUNG RADIALLY OUTWARD (BY CENTRIFUGAL FORCE). BY CONT'Y, THE FLUID FLUNG OUTWARD IS REPLACED BY A VERTICALLY FLUID MOVING TOWARD THE CYLINDER ENDS



10/16/99 (3)

BOUNDARY LAYER SCALES

A. BOUNDARY LAYER IS CREATED NEAR THE SPINNING DISK — VORTICITY GENERATED AT THE DISK BOUNDARY IS CONFINED TO THE NEAR-DISK REGION BY THE DOWNWARD FLUID MOTION.

THE CHARACTERISTIC BOUNDARY LAYER THICKNESS, $\hat{\delta}$, THE VERTICAL VELOCITY SCALE, $\hat{w}_s$, AND THE RADIAL VELOCITY SCALE, $\hat{u}_s$, ARE DETERMINED SIMULTANEOUSLY USING THE FOLLOWING ARGUMENTS:

1) THE DOMINANT VISCOUS FORCE (W/IN B.L.) MUST BE OF SAME ORDER AS THE CENTRIFUGAL FORCE DRIVING OUTWARD FLOW:

$$\hat{\mu}\hat{u}_{,\hat{z}\hat{z}} \approx \hat{\rho}\frac{\hat{v}^2}{\hat{r}}$$

$$\Rightarrow \boxed{\frac{\hat{\mu}\hat{u}_s}{\hat{\delta}^2} \approx \hat{\rho}\frac{\hat{v}_s^2}{\hat{a}}} \qquad (A)$$

NOTE: THE AZIMUTHAL VELOCITY SCALE IS

$$\boxed{\hat{v}_s = \hat{\Omega}_1\hat{a}} \qquad (B)$$

2) THE VERTICAL VELOCITY SCALE, $\hat{w}_s$, IS (AS USUAL) DETERMINED BY CONTINUITY:

$$rw_{,z} + (ru)_{,r} = 0$$

$$\Rightarrow \frac{\hat{w}_s}{\hat{\delta}} \approx \frac{\hat{u}_s}{\hat{a}}.$$

$$\Rightarrow \boxed{\hat{w}_s \approx \frac{\hat{\delta}}{\hat{a}}\hat{u}_s} \qquad (C)$$

10/16/99 (3A)

3) DIFFUSION OF VORTICITY AWAY FROM THE WALL IS CONFINED NEAR THE WALL BY THE DOWNWARD/UPWARD VELOCITY TOWARD THE END WALLS.

FROM THE r-VORTICITY EQN, WE THUS BALANCE VERTICAL ADVECTION OF r-VORTICITY AGAINST THE DOMINANT DIFFUSION TERM:

$$w\,\omega_{r,z} \approx \nu\,\omega_{r,zz} \qquad \left(\omega_r = (\nabla \times \underline{u})_r\right)$$

$$\Rightarrow \frac{\hat{w}_s}{\hat{\delta}} \approx \frac{\hat{\nu}}{\hat{\delta}^2}$$

$$\Rightarrow \boxed{\hat{w}_s = \frac{\hat{\nu}}{\hat{\delta}}} \qquad (D)$$

NOW SOLVE (A), (B) & (C) FOR $\hat{w}_s$, $\hat{\delta}$, AND $\hat{u}_s$:

(C) & (D) $\Rightarrow$

$$\hat{w}_s = \frac{\hat{\nu}}{\hat{\delta}} = \frac{\hat{\delta}}{\hat{a}}\hat{u}_s$$

$$\Rightarrow \frac{\hat{\nu}}{\hat{\delta}^2} = \frac{\hat{u}_s}{\hat{a}} \qquad (E)$$

(A) + (B) $\Rightarrow$

$$\frac{\hat{\nu}}{\hat{\delta}^2} = \frac{\hat{V}_s^2}{\hat{a}\hat{u}_s} = \frac{\hat{u}_s}{\hat{a}}$$

$$\Rightarrow \boxed{\hat{u}_s = \hat{V}_s = \hat{\Omega}_1 \hat{a}} \qquad (F)$$

(F) + (E) $\Rightarrow$

$$\boxed{\hat{\delta} = \sqrt{\frac{\hat{\nu}\hat{a}}{\hat{u}_s}} = \sqrt{\frac{\hat{\nu}\hat{a}}{\hat{\Omega}_1 \hat{a}}} = \sqrt{\frac{\hat{\nu}}{\hat{\Omega}_1}}} \qquad (G)$$

NOTE, PRESSURE SCALE IS ~~DETERMINED BY~~ $\boxed{\hat{p} = \hat{\rho}\hat{V}_s^2 = \hat{\rho}\hat{\Omega}_1^2\hat{a}^2}$ (H)

10/16/99 3B

NOTE, IN DERIVING (D), (F) & (G) WE ASSUMED AN $\hat{r}$ LENGTH SCALE OF $\hat{a}$ AND VERTICAL ($\hat{z}$) LENGTH SCALE OF $\hat{\delta}$.

$$\Rightarrow \boxed{(\hat{r}_s, \hat{z}_s) = (\hat{a}, \hat{\delta}) = \left(\hat{a}, \sqrt{\frac{\nu}{\hat{\Omega}_1}}\right)} \quad (I)$$

NOTE, WE'LL SHOW LATER THAT THE BOUNDARY LAYER TIME SCALE IS

$$\boxed{\hat{t}_{BL} = \hat{\Omega}_1^{-1}}$$

(SEE PAGES 8 - 10 FOR SCALES IN CORE REGION)

11/26/89 (4)

THUS, WE ASSUME BOUNDARY LAYERS COAT UPPER AND LOWER BOUNDARIES, THE FLUID WILL ASSUME THE NEW ROTATION RATE ONCE ALL OF IT HAS BEEN 'PUMPED' THROUGH THESE END BOUNDARY LAYERS, IN OTHER WORDS

$$\boxed{\hat{t}_{adjustment} \equiv \hat{t}_a \approx \frac{\text{vol fluid}}{\text{vol. flow rate into end Boundary layers}} \approx \frac{\hat{L}\hat{a}^2}{\hat{Q}}} \qquad (3)$$

WHERE Q can BE ESTIMATED AS THE vol. flow RATE THROUGH ONE OF THE END BOUNDARY LAYERS:

$$\hat{Q} \approx \hat{u}_s \hat{\delta} \hat{a} \approx \hat{u}_s \left(\frac{\nu}{\Omega}\right)^{1/2} \hat{a} \qquad (4)$$

FROM (F), $\hat{u}_s = \hat{\Omega}_1 \hat{a}$

$$\hat{Q} \approx \hat{\Omega}_1 \hat{a}^2 \hat{\delta}$$

$$\Rightarrow \boxed{\hat{t}_a = \frac{\hat{L}\hat{a}^2}{\hat{\Omega}_1 \hat{a}^2 \hat{\delta}} = \Omega_1^{-1} \epsilon^{-1}} \qquad (5)$$

WHERE $\epsilon \equiv \delta / L$ = EKMAN NO.,

(NEXT PARAGRAPH REPEATS (5))

10/16/94

(5)

USING (5) IN (4) AND INSERTING THAT RESULT IN (3) GIVES:

$$Q \approx a\Omega\left(\frac{\nu}{\Omega}\right)^{1/2} a$$

$$\Rightarrow \boxed{t_a \approx \frac{La^2}{a^2\Omega\sqrt{\nu/\Omega}} = \frac{L}{\Omega\delta} = \frac{1}{\Omega\epsilon}} \quad (6)$$

WHERE $\boxed{\epsilon \equiv \delta/L = \frac{\sqrt{\nu/\Omega}}{L} = \text{EKMAN NUMBER}}$ (7)

* THE TIME SCALE FOR THIS PROBLEM IS t_a, GIVEN IN (6).

NOTE, IF THE CYLINDER WERE INFINITELY LONG, THEN THE FLOW WOULD ADJUST TO THE NEW ROTATION RATE BY DIFFUSION OF VORTICITY (CREATED AT FASTER WALLS) ACROSS CYLINDER RADIUS. THE TIME SCALE IN THIS CASE IS OBTAINED BY BALANCING THE TIME DERIVATIVE OF VORTICITY AGAINST THE DOMINANT DIFFUSION TERM:

FROM vertical vorticity eqn:

$$\omega_{,t} \approx \nu\omega_{,rr} \Rightarrow \frac{1}{t_{a'}} \approx \frac{\nu}{a^2} \Rightarrow \boxed{t_{a'} \approx a^2/\nu} \quad (8)$$

10/16/99 (6)

STATEMENT OF IBVP (INITIAL BOUNDARY VALUE PROBLEM) $\left(\frac{\partial}{\partial\theta}=0\right)$

θ: $v_{,t} + u\left(v_{,r} + \frac{v}{r}\right) + wv_{,z} = \nu\left(v_{,rr} + \frac{v_{,r}}{r} + v_{,zz}\right)$ (9a)

r: $u_{,t} + uu_{,r} + wu_{,z} - \frac{v^2}{r} = \nu\left(u_{,rr} + \left(\frac{u}{r}\right)_{,r} + u_{,zz}\right) - \frac{1}{\rho}p_{,r}$ (9b)

z: $w_{,t} + uw_{,r} + ww_{,z} = -\frac{1}{\rho}p_{,z} + \nu\left(w_{,rr} + \frac{w_{,r}}{r} + w_{,zz}\right)$ (9c)

CONT'Y $\quad (ru)_{,r} + rw_{,z} = 0$ (9d)

B.C.'s $\boxed{t>0}$ ON

$z = 0, L \quad u = w = 0 \quad v = r\Omega_1$ (9e)

$r = a \quad u = w = 0 \quad v = a\Omega_1$ (9f)

$r = 0 \quad u = v = 0$ (9g)

I.C. $\quad t<0$

$u = w = 0$

$v = \Omega_0 r$ (9h)

NOTE $\quad \Omega_1 = \Omega_0 + \Delta\Omega \Rightarrow \Delta\Omega = \Omega_1 - \Omega_0$

10/10/99 (1)

$\boxed{\text{NONDIMENSIONALIZATION}}$ (use carets to denote dim'l qtys.)

$$t = \hat{t}/\hat{t}_0 = \hat{t}\hat{\Omega}_1\epsilon$$

$$p = \hat{p}/\hat{p}_s = \hat{p}/\hat{\rho}\hat{\Omega}_1^2\hat{a}^2$$

$$\boxed{\hat{p}_{,r} \approx \hat{\rho}\frac{\hat{v}^2}{\hat{r}} \approx \hat{\rho}\frac{\hat{\Omega}_1^2\hat{a}^2}{\hat{a}}} \Rightarrow \boxed{\hat{p}_s \approx \hat{\rho}\hat{\Omega}_1^2\hat{a}^2}$$

$$(r, z) = (\hat{r}, \hat{z})/\hat{a}$$

$$(u, v, w) = (\hat{u}, \hat{v}, \hat{w})/\hat{\Omega}_1\hat{a}$$

let $\hat{V}_s = \hat{\Omega}_1 \hat{a}$

$$\boxed{\epsilon_0 = \hat{a}/\hat{L} \approx O(1)}$$

ASSUME $\epsilon_0 \gg \epsilon$

$\Rightarrow$ (9a) – (9d) BECOME

$\underline{\underline{\theta}}$: $\hat{\Omega}_1\epsilon v_{,t} + \frac{\hat{V}_s \hat{\Omega}_1}{\hat{a}}[u(v_{,r} + \frac{v}{r})] = \frac{\hat{\nu}}{\hat{a}^2}[v_{,zz} + v_{,rr} + (\frac{v}{r})_{,r}]$

$$\boxed{\epsilon v_{,t} + u(v_{,r} + \frac{v}{r}) + wv_{,z} = \cancel{\frac{\hat{L}^2 \hat{a}^2}{\hat{L}^2\hat{a}^2}}^{\epsilon^2/\epsilon_0^2}[v_{,zz} + v_{,rr} + (\frac{v}{r})_{,r}]} \quad (10a)$$

$\underline{\underline{r}}$: $\hat{\Omega}_1\epsilon u_{,t} + \frac{\hat{V}_s \hat{\Omega}_1 \hat{a}}{\hat{a}}(uu_{,r} + wu_{,z}) - \frac{\hat{V}_s}{\hat{a}}\frac{v^2}{r} = -\frac{\hat{\rho}\hat{\Omega}_1^2\hat{a}^2}{\hat{V}_s\hat{\rho}\hat{a}}p_{,r} + \frac{\hat{\nu}}{\hat{a}^2}[u_{,rr} + (\frac{u}{r})_{,r} + u_{,zz}]$

$$\Rightarrow \boxed{\epsilon u_{,t} + uu_{,r} + wu_{,z} - \frac{v^2}{r} = -p_{,r} + \frac{\epsilon^2}{\epsilon_0^2}[u_{,rr} + (\frac{u}{r})_{,r} + u_{,zz}]} \quad (10b)$$

$\underline{\underline{z}}$:

$$\boxed{\epsilon w_{,t} + uw_{,r} + ww_{,z} = -p_{,z} + \frac{\epsilon^2}{\epsilon_0^2}[w_{,rr} + w_{,zz} + \frac{w_{,r}}{r}]} \quad (10c)$$

$\underline{\underline{\text{CONT'N}}}$: $\boxed{(ru)_{,r} + rw_{,z} = 0}$ (10d)

DIM'LESS

B.C.'s :

$$z = 0, 1/\epsilon_0 \; : \quad u = w = 0 \qquad v = r \tag{10e}$$

$$r = 1 \; : \quad u = w = 0 \qquad v = 1 \tag{10f}$$

$$r = 0 \; : \quad u = v = 0 \tag{10g}$$

I.C.

$$u = w = 0$$

$$v = \frac{\Omega_0}{\Omega_1} r \tag{10h}$$

10/16/99 (8)

WE WILL BREAK THE PROBLEM/SOLN INTO 2 PARTS: 1 PORTION APPLIES TO THE CORE REGION (WHERE VORTICITY FROM THE UPPER/LOWER BOUNDARIES HAS NOT REACHED) AND 1 PORTION WHICH APPLIES IN THE END BOUNDARY LAYERS.

$\boxed{\text{CORE REGION}}$

IN CORE, WE EXPECT THAT THE u AND w VELOCITY COMPONENTS ARE NEGLIGIBLE COMPARED TO THE v COMPONENT. PRIOR TO ASSUMING AN ASYMPTOTIC FORM FOR THE CORE FLOW, LET'S ESTIMATE THE MAGNITUDES OF THE r and z vel. COMPONENTS W/IN THE CORE. THE z COMP. CAN BE ESTIMATED BY NOTING THAT THE UPWARD (OR DOWNWARD) vel's NEAR z = 0, L ARE APPROXIMATELY EQUAL TO THE VERTICAL velocity SCALE $\hat{w}_3$ W/IN THE END BOUNDARY

10/10/99 (9)

LAYERS:

$$\underline{\hat{W}}_s \approx \hat{w}_s$$

(use underscores for core qty's)

But from cont'y w/in boundary layers

$$\frac{\hat{w}_s}{\hat{\delta}} \approx \frac{\hat{u}_s}{\hat{a}} \Rightarrow \hat{w}_s \approx \hat{u}_s \frac{\hat{\delta}}{\hat{a}} \approx \hat{\Omega}_1 \hat{a} \frac{\hat{\delta}}{\hat{a}}$$

$$\boxed{\underline{\hat{W}}_s \approx \hat{w}_s \approx \hat{\Omega}_1 \hat{\delta}} \qquad (11)$$

Now, we can estimate r-vel. scale $\underline{\hat{u}}_s$ w/in core from cont'y:

$$\frac{\underline{\hat{u}}_s}{\hat{a}} \approx \frac{\underline{\hat{w}}_s}{\hat{L}} \Rightarrow \boxed{\underline{\hat{u}}_s \approx \underline{\hat{w}}_s \epsilon_0 = \epsilon_0 \hat{\Omega}_1 \hat{\delta}} \qquad (12)$$

$$\left(\epsilon_0 = \hat{a}/\hat{L} = O(1)\right)$$

Thus,

$$\boxed{\frac{\underline{\hat{u}}_s}{\underline{\hat{V}}_s} \approx \frac{\epsilon_0 \hat{\Omega}_1 \hat{\delta}}{\hat{\Omega}_1 \hat{a}} \approx \frac{\hat{\delta}}{\hat{a}} \ll 1} \quad (\text{since } \epsilon_0 = O(1)) \qquad (13)$$

and

$$\boxed{\frac{\underline{\hat{w}}_s}{\underline{\hat{V}}_s} \approx \frac{\hat{\Omega}_1 \hat{\delta}}{\hat{\Omega}_1 \hat{a}} = \frac{\hat{\delta}}{\hat{a}} \ll 1} \qquad (14)$$

10/16/94 (10)

SINCE $\epsilon_0 = \hat{a}/L = O(1)$, (13) & (14) IMPLY THAT

$$\boxed{\begin{aligned}\hat{\underline{U}}_s &= \epsilon \hat{\underline{V}}_s \\ \hat{\underline{W}}_s &= \epsilon \hat{\underline{V}}_s\end{aligned}} \qquad (15)$$

NOTE $\boxed{\text{W/IN CORE}}$ $\hat{P}_s = \rho \underline{V}_s^2 = \rho \Omega_1^2 a_1^2$,

THUS, LET'S ASSUME AN ASYMPTOTIC SOLN. W/IN THE CORE OF FOLLOWING FORM:

$\boxed{\text{CORE SOLN}}$

$$U = 0 + \epsilon U_1(r,z,t) \qquad (16a)$$

$$V = V_0(r,z,t) + \epsilon V_1(r,z,t) \qquad (16b)$$

$$W = 0 + \epsilon W_1(r,z,t) \qquad (16c)$$

$$P = P_0(r,z,t) + \epsilon P_1(r,z,t) \qquad (16d)$$

INSERTING (16)(a-c) INTO (10a)-(10d) WE OBTAIN THE FOLLOWING PROBLEMS AT $O(\epsilon^0) = O(1)$ and $O(\epsilon)$: (NOTE $\epsilon^2/\epsilon_0^2 = O(\epsilon^2)$)

$O(1)$:

$\underline{\underline{r}}$: $\boxed{-\frac{V_0^2}{r} = -P_{0,r}}$ (17a)

$\underline{\underline{\theta}}$: no order 1 terms

$\underline{\underline{z}}$: $\boxed{P_{0,z} = 0}$ (17b)

CONT'Y NO order 1 TERMS

$$\therefore (17a)_{,z} \Rightarrow \quad V_{0,z} = 0 \quad \Rightarrow \quad \boxed{V_0 = V_0(r,t)} \qquad (18)$$

$O(\varepsilon)$:

$$\underline{\underline{\theta}}: \quad V_{0,t} + u_1\left(V_{0,r} + \frac{V_0}{r}\right) + w_1 \overset{0 \text{ BY } (18)}{\cancel{V_{0,z}}} = 0 \qquad (19a)$$

$$\underline{\underline{r}}: \quad -\frac{2V_0 V_1}{r} = -P_{1,r} \qquad (19b)$$

$$\underline{\underline{z}}: \quad 0 = -P_{1,z} \qquad (19c)$$

$$\underline{\underline{\text{CONT'Y}}} \quad (rU_1)_{,r} + (rW_1)_{,z} = 0 \qquad (19d)$$

PRIOR TO ATTEMPTING A SOLN FOR THE <u>CORE</u> <u>FLOW</u>, LET'S PUT DOWN THE $O(1)$ & $O(\varepsilon)$ PROBLEMS W/IN THE END BOUNDARY LAYERS:

FIRST, WE HAVE TO CHANGE THE LENGTH SCALE IN THE Z-DIRECTION (TO $\hat{\delta}$). THIS HAS THE EFFECT OF AMPLIFYING DERIVATIVES W.R.T. Z. <u>PHYSICALLY</u>, WE ONCE AGAIN RECOGNIZE THAT W/IN END B.L.'S, THE DOMINANT VISCOUS FORCE WILL BE COMPARABLE TO THE DOMINANT INERTIAL FORCE. [IN THEIR PRESENT DIM'LESS FORM, THE GOVERNING EQNS (IN (10a)-(10d))

10/12/99 (11a)

NOTE THAT WE COULD HAVE OBTAINED THE SAME SETS OF EQNS AT $O(1)$ AND $O(\epsilon)$. BY INITIALLY USING THE FOLLOWING VELOCITY SCALES FOR u & w AND v:

$$\hat{u}_s = \epsilon \hat{V}_s$$
$$\hat{w}_s = \epsilon \hat{V}_s$$
$$\hat{V}_s = \hat{\Omega}_1 \hat{a}$$
$$P_s = \hat{\rho}\hat{\Omega}_1^2 \hat{a}^2$$

AND THEN BY ASSUMING AN ASYMPTOTIC SOLN OF FORM:

$$\boxed{\begin{aligned} u &= u_0 + \epsilon_1 u_1 \\ v &= v_0 + \epsilon_1 v_1 \\ w &= w_0 + \epsilon_1 w_1 \\ P &= P_0 + \epsilon_1 P_1 \end{aligned}} \quad \begin{aligned} &(20a) \\ &(20b) \\ &(20c) \\ &(20d) \end{aligned}$$

IN THIS APPROACH, u_0 replaces u_1 AND w_0 replaces w_1 IN ORIGINAL APPROACH. KEY IS TO GET PROPER SCALES ON DIM'L VELOCITYS $\hat{u}_s$, $\hat{w}_s$ AND $\hat{V}_s$.

$O(1)$:

θ: NO ORDER 1 TERMS

r: $-\frac{v_0^2}{r} = -P_{0,r}$

z: $0 = P_{0,z}$ CONT'Y: NO $O(1)$ TERMS

$O(\epsilon)$:

θ: $v_{0,t} + u_0\left(v_{0,r} + \frac{v_0}{r}\right) + w_0 v_{0,z} = 0$

r: $-2\frac{v_0 v_1}{r} = P_{1,r}$ | z: $0 = -P_{1,z}$ | CONT'Y: $(ru_0)_{,r} + (rw_0)_{,r} = 0$

10/16/99 (12)

WHICH APPLY W/IN THE CORE, SHOW THAT VISCOUS TERMS ARE $O(\varepsilon^2)$ RELATIVE TO 1ST ORDER CONVECTION TERMS. THIS SCALING CANNOT HOLD W/ BOUNDARY LAYERS.] WITHIN WIND BOUNDARY LAYERS, THE PROPER Z-SCALE IS $\hat{\delta}$:

$$\boxed{z = \hat{z}/\hat{\delta}} \qquad \hat{\delta} = \sqrt{\hat{\nu}/\hat{\Omega}} \qquad (20)$$

OR COMPARING DIM'LESS z W/IN CORE TO DIM'LESS z W/IN BOUND. LAYERS:

$$z_{core} = \frac{\hat{z}}{\hat{a}} \qquad z_{B.L.} = \hat{z}/\hat{\delta} = \frac{\hat{z}}{\hat{a}}\,\hat{a}/\hat{\delta}$$

$$= z_{core}\,\frac{\hat{a}}{\hat{L}}\,\frac{\hat{L}}{\hat{\delta}}$$

$$= z_{core}\,\epsilon_0/\epsilon$$

$$\Rightarrow \boxed{z_{BL} = z_{core}\,\frac{\epsilon_0}{\epsilon}} \qquad (21)$$

LET $\boxed{Z = \frac{\epsilon_0}{\epsilon}\, z}$ (22)

SIMPLEST WAY TO GET RESCALED EQNS W/IN B-LAYERS IS TO REPLACE z IN (10a)–(10d) W/ Z FROM (22)

10/16/99 (13)

BOUNDARY LAYER

(10a) $\Rightarrow$

$\underline{\underline{\theta}}$: $\epsilon v_{,t} + u\left(v_{,r} + \frac{v}{r}\right) + \frac{\epsilon_0}{\epsilon} w v_{,\underline{z}} = v_{,\underline{z}\underline{z}} + \frac{\epsilon^2}{\epsilon_0^2}\left[v_{,rr} + \left(\frac{v}{r}\right)_{,r}\right]$ (23a)

$\underline{\underline{r}}$: $\epsilon u_{,t} + u u_{,r} + \frac{\epsilon_0}{\epsilon} w u_{,\underline{z}} = - p_{,r} + u_{,\underline{z}\underline{z}} + \frac{\epsilon^2}{\epsilon_0^2}\left[u_{,rr} + \left(\frac{u}{r}\right)_{,r}\right]$ (23b)

: $\epsilon w_{,t} + u w_{,r} + \frac{\epsilon_0}{\epsilon} w w_{,\underline{z}} = -\frac{\epsilon_0}{\epsilon} p_{,\underline{z}} + w_{,\underline{z}\underline{z}} + \frac{\epsilon^2}{\epsilon_0^2}\left[w_{,rr} + \frac{w_{,r}}{r}\right]$ (23c)

CONTY: $\boxed{(ru)_{,r} + \frac{\epsilon_0}{\epsilon}(rw)_{,\underline{z}} = 0}$ (23d)

NOW, WE ASSUME FORM OF ASYMPTOTIC BOUNDARY LAYER VELOCITY FIELD. SINCE WE USED ... THE SAME O(1) VELOCITY SCALE TO NON-DIMENSIONALIZE THE GOVERNING EQNS. [i.e., WE USED $\hat{U}_s = \hat{V}_s = \hat{W}_s = \hat{\Omega}\hat{a}$ TO ARRIVE AT (10a)–(10d)], THEN JUST AS IN THE CASE OF THE ASSUMED CORE SOLN., (16a)–(16d), WE MUST TAKE PROPER ACCT. OF THE RELATIVE SIZES OF u, v, w W/IN BOUNDARY LAYERS. AS ALREADY NOTED, THE FOLLOWING SCALES APPLY W/IN END B. LAYERS:

$$\hat{u}_s = \hat{\Omega}_1 \hat{a}$$
$$\hat{v}_s = \hat{\Omega}_1 \hat{a}$$
$$\hat{w}_s = \frac{\hat{u}_s}{\hat{a}} \hat{\delta} \quad \text{(FROM CONT'Y)}$$
$$= \frac{\hat{\Omega}_1 \hat{a}}{\hat{a}} \hat{\delta} = \hat{\Omega}_1 \hat{\delta} \qquad \hat{\delta} = \sqrt{\hat{\nu}/\hat{\Omega}_1}$$
$$\hat{P}_s = \hat{\rho} \hat{\Omega}_1^2 \hat{a}^2$$

NOW COMPARING $\hat{u}_s$, $\hat{v}_s$ & $\hat{w}_s$, WE SEE THAT

$$\boxed{\begin{aligned} \frac{\hat{u}_s}{\hat{v}_s} &= O(1) \\ \frac{\hat{w}_s}{\hat{v}_s} &= \frac{\hat{\delta}}{\hat{a}} = \frac{\hat{\delta}}{\hat{a}} \frac{\hat{L}}{\hat{L}} = \epsilon/\epsilon_0 \end{aligned}} \qquad (24)$$

THUS, WE ASSUME THE FOLLOWING ASYMPTOTIC SOLN. W/IN END BOUNDARY LAYERS:

$$\boxed{\begin{aligned} u &= u_0(r,z,t) + \epsilon u_1(r,z,t) && (25a) \\ v &= v_0(r,z,t) + \epsilon v_1(r,z,t) && (25b) \\ w &= 0 + \epsilon w_1(r,z,t) && (25c) \\ P &= P_0(r,z,t) + \epsilon P_1(r,z,t) && (25d) \end{aligned}}$$

10/14/99 (15)

NOW, INSERT (25a) – (25d) INTO GOVERNING EQNS (23a) – (23d):

O(1)

$\underline{\theta}$: $u_0\left(V_{0,r} + \frac{V_0}{r}\right) + \epsilon_0 w_1 V_{0,\underline{z}} = V_{0,\underline{zz}}$ (26a)

$\underline{r}$: $u_0 u_{0,r} + w_1 u_{0,\underline{z}} - \frac{V_0^2}{r} = -P_{0,r} + u_{0,\underline{zz}}$ (26b)

$\underline{z}$: $P_{0,\underline{z}} = 0$ (NOTE, WE MULTIPLY (23c) BY ϵ TO GET THIS RESULT) (26c)

CONT'Y: $(r u_0)_{,r} + (r w_1)_{,\underline{z}} = 0$ (26d)

B.C.'s ON $\underline{z}=0$ / $O(1)$:

$\underline{z}=0$ $u_0 = W_0 = 0$ (26e)

$\underline{z}=0$ $V_0 = r$ (26f)

$\underline{z}\to\infty$ $u_0 \to u_{0_{core}} = 0$

$V_0 \to V_{0_{core}} = \underline{V_0}(r,t)$ (26g)

NOTE, WE ASSUME THAT THE FLOW W/IN THE B. LAYERS IS QUASISTEADY. THIS ASSUMPTION IS VALID SINCE THE TIME SCALE FOR VELOCITY CHANGES W/IN B.L.'S IS Ω_1^{-1}. ONE WAY TO SEE THIS IS TO ESTIMATE THE

10/16/99 16

TIME RATE OF CHANGE OF (RADIAL) VORTICITY AGAINST THE DOMINANT VORTICITY DIFFUSION TERM (SINCE DIFFUSION OF VORTICITY IS IMPORTANT W/IN B. LAYERS):

$$\hat{\omega}_{,\hat{t}} \approx \hat{\nu}\,\hat{\omega}_{,\hat{z}\hat{z}} \qquad \underline{\text{DIMENSIONAL}}$$

$$\Rightarrow \frac{1}{\hat{t}_s} \approx \hat{\nu}/\hat{\delta}^2 -$$

$$\Rightarrow \boxed{\hat{t}_s = \frac{\hat{\delta}^2}{\hat{\nu}} \approx \frac{(\hat{\nu}/\hat{\Omega}_1)}{(\hat{\nu})} = \hat{\Omega}_1^{-1}} \qquad (22)$$

COULD OBTAIN SAME ESTIMATE BY BALANCING TIME RATE OF Δ OF $\hat{V}$ W/ DOMINANT VISCOUS FORCE (IN θ-MOMENTUM EQN.):

$$\hat{V}_{,\hat{t}} \approx \hat{\nu}\,\hat{V}_{,\hat{z}\hat{z}}$$

$$\frac{1}{\hat{t}_s} \approx \hat{\nu}/\hat{\delta}^2 \Rightarrow \underline{\hat{t}_s \approx \hat{\Omega}_1^{-1}}$$

THUS, INITIAL TIME VARIATIONS (INDUCED BY CHANGING Ω TO $\Omega_0 + \Delta\Omega$) IN u, v, w OR p W/IN B. LAYER HAPPENS <u>QUICKLY</u> RELATIVE TO ADJUSTMENT TIME-SCALE $\hat{t}_a \left(\in \frac{1}{\hat{\Omega}_1 \epsilon}\right)$:

$$\boxed{\frac{\hat{t}_s}{\hat{t}_a} \approx \frac{\hat{\Omega}_1^{-1}}{\hat{\Omega}_1^{-1}\epsilon^{-1}} = \epsilon \ll 1}$$

10/6/99 (7)

HENCE, ANY TIME VARIATIONS W/IN CORE OCCUR ON A LONG TIME SCALE RELATIVE TO TIME VARIATIONS IN B. LAYERS.

THIS IS EQUIVALENT TO ASSUMING QUASISTATIC CONDITIONS W/IN B. LAYERS.

(E.G., - IF WE MODEL FLOW AROUND A STEADILY MOVING SHIP ON THE OCEAN, WE NEGLECT THE EFFECT OF <u>SLOW</u> TIDE VARIATIONS ON FLOW - I.E., WE ASSUME FLOW ABOUT SHIP IS STEADY.)

IN GENERAL, WE WOULD HAVE TO SOLVE THE CORE EQNS AND B. LAYER EQNS IN (17), (19) AND (26) NUMERICALLY (SINCE THE SIMPLIFIED SYSTEMS ARE STILL NON-LINEAR). HOWEVER, WE CAN OBTAIN A SOLUTION IN THE CASE WHERE

$$\boxed{\frac{\Delta\Omega}{\Omega_0} = \frac{\Omega_1 - \Omega_0}{\Omega_0} \ll 1.} \quad (30)$$

$$(30) \Rightarrow \frac{\Omega_1}{\Omega_0} - 1 \ll 1 \Rightarrow \frac{\Omega_1}{\Omega_0} \approx 1$$

FROM (26c), $P_{0,r}$ DOES NOT DEPEND ON z. THUS, WE CAN DETERMINE $P_{0,r}$ BASED ON IT'S MAGNITUDE OUTSIDE B. LAYER, i.e., AS $z \to \infty$.

IN (26b), when $z \to \infty$, $u_0 \to 0$ AND $v_0 \to \underline{V_0}(r,t)$, THUS, FROM (26b)

$$(31) \quad \boxed{P_{0,r} = \frac{\underline{V_0}^2}{r}} \qquad (\underline{V_0} = \theta\text{-vel w/in core})$$

THUS, (26b) W/IN B. LAYER CAN BE WRITTEN AS

$$\underline{\underline{r}}: \quad \boxed{u_0 u_{0,r} + w_1 u_{0,z} + \frac{1}{r}\left(\underline{V_0}^2 - V_0^2\right) = u_{0,zz}} \quad (26b')$$

IN ORDER DEVELOP A SOLN., WE ASSUME THAT FOR SMALL CHANGES IN ROTATION RATE, THE ADJUSTED FLOW IS VERY SIMILAR TO THE INITIAL FLOW:

$$\boxed{\begin{array}{ll} \text{BOUNDARY LAYER:} & (u_0, v_0, w_0) = (0, r, 0) + \frac{\Delta\Omega}{\Omega_1}(\tilde{u}, \tilde{v}, \tilde{w}) \quad (31) \\ \text{CORE} & (\underline{u_0}, \underline{V_0}, \underline{w_0}) = (0, r, 0) + \frac{\Delta\Omega}{\Omega_1}(0, \underline{\tilde{V}}, 0) \quad (32) \end{array}}$$

10/14/99 (19)

NOTES: 1) WE DIVIDE $\Delta\Omega$ BY Ω_1 IN ORDER TO CANCEL Ω_1'S IN THE DIM'LESS GOVING EQNS. (23a)-(23d) AND (10a)-(10d). WE COULD JUST AS EASILY DIVIDE BY Ω_0.

2) THE ASSUMED FORM OF THE CORE SOLN. IS CONSISTENT W/ OUR EXPECTATION (GUESS) THAT THE ADJUSTED CORE FLOW (AT LEAST AT LEADING ORDER IN $\Delta\Omega/\Omega_1$) WILL BE VERY SIMILAR TO THE SOLID BODY ROTATIONAL FLOW THAT EXISTS INITIALLY. IF WE CAN SATISFY ALL B.C.'S W/ THE ASSUMED SOLNS. IN (31) & (32), THEN THESE SOLNS. ARE AT LEAST PLAUSIBLE. WOULD HAVE TO DO EXPERIMENTS TO VERIFY OUR THEORY (GUESS). THERE WILL BE $\underline{w}$ AND $\underline{u}$ COMPONENTS, BUT WE ANTICIPATE THAT THESE WILL BE AN ORDER SMALLER (IN ε) THAN THE $O(\varepsilon)$ AZIMUTHAL velocity.

10/16/99 (20)

insert (31) into (26b') and (26a) and neglect terms of order $\left(\frac{\Delta\Omega}{\Omega_1}\right)^2$ and smaller.

LET $\epsilon_1 = \Delta\Omega/\Omega_1$

$$(26b') \Rightarrow \quad \epsilon_1^2 \tilde{u}\tilde{u}_{,r} + \omega_1 \epsilon_1 \tilde{u}_{,z} + \frac{1}{r}\left(r^2 - r^2\right) \oplus \frac{1}{r}\left(2r\epsilon_1 \tilde{V} - 2r\epsilon_1 \tilde{v}\right) = \epsilon_1 \tilde{u}_{,zz}$$

$$(26a) \Rightarrow \quad \epsilon_1 \tilde{u}(1+1) + \epsilon_0 \omega_1 \epsilon_1 \tilde{v}_{,z} = \epsilon_1 \tilde{v}_{,zz}$$

$$(26b') \Rightarrow \quad \omega_1 \tilde{u}_{,z} = \tilde{u}_{,zz} - 2(\tilde{V} - \tilde{v}) \qquad (33a)$$

$$(26a) \Rightarrow \quad 2\tilde{u} + \epsilon_0 \omega_1 \tilde{v}_{,z} = \tilde{v}_{,zz} \qquad (33b)$$

$$(26d) \Rightarrow \quad \epsilon(r\tilde{u})_{,r} + (r\omega_1)_{,z} = 0 \qquad (33c)$$

(33c) IMPLIES THAT $\omega_1 = O(\epsilon_1)$. (PHYSICALLY, WE DON'T EXPECT $\frac{\partial w_1}{\partial z} = 0$ AT LEADING ORDER. CONT'D TO SIMPLIFY TO)

WE COULD ALSO ASSUME THAT THE ADJUSTMENT TO ω_1 IS OF THE FORM

$$\omega_1 = \underset{\uparrow}{0} + \epsilon_1 \tilde{\omega}_1$$

THIS MEANS THAT THE 1ST ORDER CORRECTION TO ω_0 DURING SOLID BODY ROTATION IS 0, AS IT SHOULD BE.

THUS, BY EITHER ARGUMENT $\omega_1 = O(\epsilon_1)$

10/16/99 (21)

$\therefore$ (33a) $\Rightarrow$ $\boxed{2(\underline{\tilde{V}} - \tilde{v}) = \tilde{u}_{,zz}}$ (34a)

(33b) $\Rightarrow$ $\boxed{2\tilde{u} = \tilde{v}_{,zz}}$ (34b)

NOW DEFINE ψ AS

$$\boxed{\psi = \underline{\tilde{V}}(r,t) - \tilde{v}(r,z,t)} \quad (35)$$

$\Rightarrow$ $\tilde{v}_{,zz} = -\psi_{,zz}$

$\Rightarrow$ (34)a $\Rightarrow$ $\boxed{2\psi = \tilde{u}_{,zz}}$ (36a)

(34b) $\Rightarrow$ $\boxed{-2\tilde{u} = -\psi_{,zz}}$ (36b)

MULTIPLY (36b) by $i = \sqrt{-1}$ AND ADD result to (36a);

$$(\tilde{u} + i\psi)_{,zz} = 2(\psi - i\tilde{u}) \quad (37)$$

LET $\boxed{\phi = \tilde{u} + i\psi}$ (38)

(37) $\Rightarrow$ $\boxed{\phi_{,zz} = -2i\phi}$ (39)

B.C.'s on ϕ;

(39a) $\phi(r,t,z=0) = \overbrace{\tilde{u}(z=0,r,t)}^{0} + i\,(\underbrace{\underline{\tilde{V}}(r,t) - \tilde{v}(z=0,r,t)}_{})$

$\qquad = i\,\underline{\tilde{V}}(r,t)$

(39b) $\phi(r,t,z\to\infty) = \overbrace{\tilde{u}(z\to\infty,r,t)}^{0} + i\,\underbrace{(\underline{\tilde{V}}(r,t) - \tilde{V}(r,z\to\infty,t))}_{0}$

$\qquad = 0$

10/18/99 (22)

soln (39) $\Rightarrow$

$$\Phi_{,zz} - m^2\Phi = 0 \qquad m^2 = -2i$$

$$\Rightarrow m = \pm i(2i)^{1/2} = \pm\sqrt{2}\, i^{3/2} \qquad i^{3/2} = e^{i3\pi/4} = -\tfrac{1}{\sqrt{2}} + \tfrac{i}{\sqrt{2}}$$

$$\Rightarrow \quad \Phi = a_1 e^{\sqrt{2} i^{3/2} z} + b_1 e^{-\sqrt{2} i^{3/2} z} + \cancel{f(r,t)}^{\,0}$$

$$\Phi(z=0, r, t) = i\underline{\tilde{V}}(r,t) = a_1 + b_1$$

$$\Phi(z\to\infty, r, t) = 0 = a_1(0) + b_1(\infty) \Rightarrow b_1 = 0$$

$$\Rightarrow a_1 = i\underline{\tilde{V}}$$

$$\Rightarrow \underline{\Phi} = i\underline{\tilde{V}} e^{(-z + iz)}$$

$$\boxed{\Phi = i\underline{\tilde{V}} e^{-z}\left(\cos(z) + i\sin(z)\right)} \qquad (40)$$

$$\Rightarrow \tilde{u} = Re(\Phi)$$

(*) $\boxed{\tilde{u} = -\underline{\tilde{V}}(r,t)\, e^{-z}\sin(z)}$

$$\tilde{\psi} = Im(\Phi) = \underline{\tilde{V}}\, e^{-z}\cos(z) = \underline{\tilde{V}} - \tilde{v} \qquad (41)$$

(*) $\Rightarrow \boxed{\tilde{v} = \underline{\tilde{V}}(r,t)\left[1 - e^{-z}\cos(z)\right]}$ (42)

IN ORDER TO FINISH SOON, WE FIRST NOTE THAT (19a), WHICH APPLIES IN THE CORE, CAN BE EXPRESSED AS

$$\boxed{\Gamma_{,t} + \underline{u}_1 \Gamma_{,r} = 0} \tag{43}$$

WHERE $\boxed{\Gamma = r\underline{V}_\theta(r,t)}$ (V_θ = azimuthal vel. in core)

($\underline{u}_1$ = radial core vel. @ $O(\varepsilon)$)

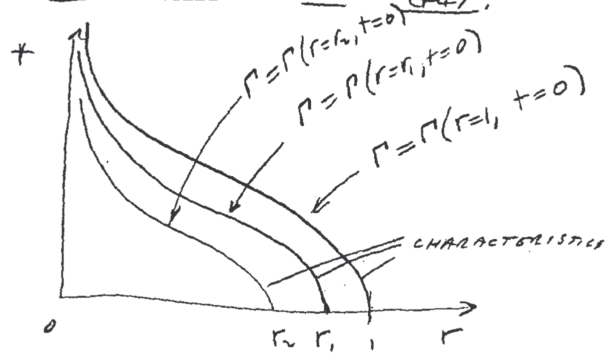
[NOTE, BY $\frac{\partial}{\partial z}$(19a), WE FIND THAT $u_1 \neq fn(z) \Rightarrow u_1 = u_1(r,t)$.]. EQN. (43) CAN BE EXPRESSED IN <u>CHARACTERISTIC</u> FORM:

$$\frac{d\Gamma}{dt} = \Gamma_{,t} + \Gamma_{,r}\frac{dr}{dt}^{\nearrow \underline{u}_1(r,t)} = 0 \tag{44}$$

IN WORDS, (44) STATES THAT "on the (r,t) plane, Γ IS constant on a <u>curve</u> (i.e., a <u>characteristic</u>) given by

$$\frac{dr}{dt} = \underline{u}_1(r,t) \text{''} \tag{45}$$

<u>PICTORIAL</u> <u>INTERPRETATION</u> <u>OF</u> <u>(44)</u>:



NOTE: 1) $\frac{dr}{dt} \leq 0$ since $u_1 \leq 0$

2) THE VALUE OF Γ ON ANY GIVEN characteristic is = to its initial value

10/16/99 (24)

THUS, GIVEN $\underline{V_0}(r, t=0)\ (= r)$ AND $\underline{u_1}(r,t)$, WE CAN USE <u>METHOD</u> <u>OF</u> <u>CHARACTERISTICS</u> TO DETERMINE $\underline{V_0}(r,t)$ THROUGHOUT THE CORE. GIVEN $\underline{V_0}(r,t)$, WE THEN KNOW $\tilde{u}$ and $\tilde{v}$ (given by (41) & (42)) [since

$$\underline{V_0} = \frac{\Delta\Omega}{\Omega_1}\tilde{v} + r \quad (\text{BY } (32)),\ \text{i.e.,}\ \boxed{\tilde{v} = \frac{\Omega_1}{\Delta\Omega}(\underline{V_0} - r)} \quad (46)]$$

AND THE SOLN (OF BOTH THE LEADING ORDER BOUNDARY LAYER AND CORE VELOCITY FIELDS) IS COMPLETE, HENCE, TO SOLVE PROBLEM WE NEED TO DETERMINE $O(\epsilon^2)$ $\underline{u_1}(r,t)$.

IN ORDER TO FIND $u_1(r,t)$, DO A MASS BALANCE at any radius r w/in cylinder $(0 \le r \le 1)$: (WORDS: MASS FLOW RADIALLY OUTWARD FROM END BOUNDARY LAYERS = MASS FLOW INWARD WRT CORE):

$$0 = \int_0^{1/\epsilon_0} u\,dz = 2\int_0^{O(1/\sqrt{\epsilon})} u\,dz + \int_0^{1/\epsilon_0} \underline{u}\,dz$$

$$= 2\int_0^{\epsilon/\epsilon_0} u\,dz + \int_0^{1/\epsilon_0} \epsilon u_1\,dz$$

$$= 2\int_0^{\infty} u\,\epsilon\,d\underline{z} + \int_0^{1/\epsilon_0} \epsilon u_1\,dz$$

$$\underbrace{2\epsilon\int_0^{\infty} u\,d\underline{z}}_{\text{FLOW OUT OF BOUND. LAYERS}} + \underbrace{\int_0^{\hat{L}/\hat{a} = 1/\epsilon_0} \underline{u}\,dz}_{\text{FLOW INTO CORE}} = 0$$

(ASSUMES u is directed radially OUT)

10/16/99 (25)

OR
$$2\int_0^\infty (\underbrace{\cancel{u_0}}_{\text{neglect relative to } u_0} + \underbrace{\epsilon \cancel{u_1}}_{\text{FROM (25a)}})\,dz + \int_0^{Y_{E_0}} \underline{u}_1\,dz = 0$$
(FROM (31)) $\left(0 + \frac{\Delta\Omega}{\Omega_1}\tilde{u}\right)$ FROM (16a)

using (31)]
$$2\frac{\Delta\Omega}{\Omega_1}\int_0^\infty \tilde{u}\,dz = -\underline{u}_1(r,t)\left(\frac{1}{\epsilon_0}\right)$$
since $u_1 \neq fn(z)$

BUT FROM (41), FIRST INTEGRAL ABOVE BECOMES:
$$-2\frac{\Delta\Omega}{\Omega_1}\underline{\tilde{V}}(r,t)\underbrace{\int_0^\infty e^{-z}\sin(z)\,dz}_{1/2} = -\frac{1}{\epsilon_0}\underline{u}_1(r,t)$$

$$\Rightarrow \quad \underline{u}_1(r,t) = \epsilon_0\frac{\Delta\Omega}{\Omega_1}\underline{\tilde{V}}(r,t)$$
$$= \epsilon_0\frac{\cancel{\Delta\Omega}}{\cancel{\Omega_1}}\left(\frac{\cancel{\Omega_1}}{\cancel{\Delta\Omega}}\right)\left(\underline{V_0}(r,t) - r\right)$$

(48A) $$\boxed{\underline{u}_1(r,t) = \epsilon_0\left(V_0(r,t) - r\right)}$$

(48B) OR $$\boxed{\underline{u}_1(r,t) = \epsilon_0\left(\underline{V_0}(r,t) - r\right)}$$

OR IN TERMS OF $\Gamma = rV_0$
$$\boxed{\underline{u}_1(r,t) = \frac{\epsilon_0}{r}\left[\Gamma - r^2\right]} \qquad (48c)$$

INSERTING (48c) INTO (43) WE GET:

$$\Gamma_{,t} + \epsilon_0 \frac{1}{r}[\Gamma - r^2]\Gamma_{,r} = 0$$

OR

$$\boxed{\frac{d\Gamma}{dt} = 0} \qquad (49)$$

$$\text{on} \quad \frac{dr}{dt} = \frac{\epsilon_0}{r}[\Gamma - r^2] \qquad (50)$$

(50) ⇒

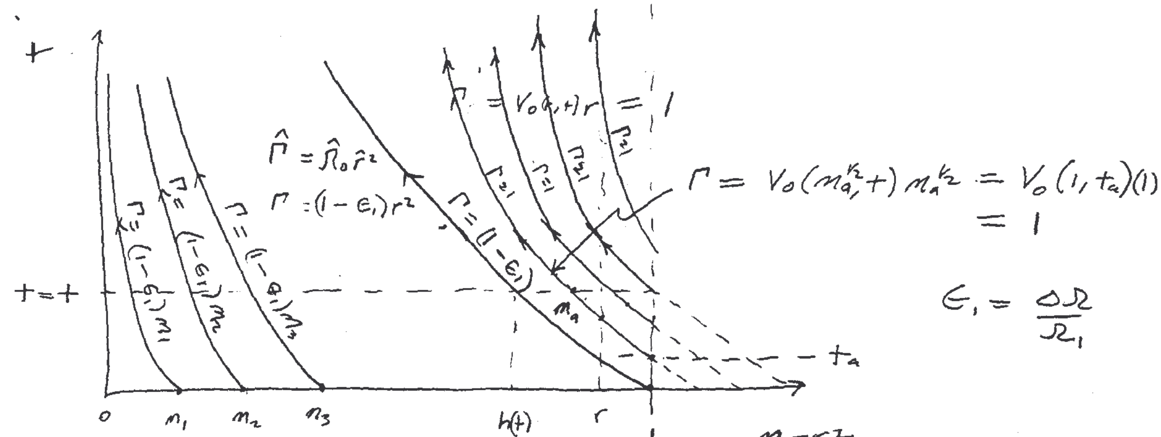
$$r\frac{dr}{dt} = \frac{1}{2}\frac{d}{dt}(r^2) = \epsilon_0[\Gamma - r^2]$$

⇒ $\boxed{\text{ON}}$

$$\boxed{\begin{aligned} \frac{d(r^2)}{dt} &= 2\epsilon_0[\Gamma - r^2] \\ \frac{d\Gamma}{dt} &= 0 \end{aligned}} \qquad \begin{aligned}(51)\\(49)\end{aligned}$$

IN WORDS, (51) & (49) STATE THAT THE MAGNITUDE OF $\Gamma = V_0(r,t)r$ REMAINS CONSTANT ON A CURVE (i.e. a characteristic) IN THE r-t plane) THAT SATISFIES THE O.D.E. (ordinary diff'l. eqn.) IN (51).

IN ORDER TO INTERPRET (51) & (49), LET'S LOOK AT THE SOLN. ON THE r-t plane:



* (at any given r (between $h(t)$ and 1) and for all times t, $\Gamma' = \Gamma(1, t') = 1 = \Gamma(r,t) = V_0(r,t) r$ $\Rightarrow \boxed{m' = 2c_0(1-m)}$ *

WE CAN SOLVE (51) & (49) FOR m coordinate @ t for the $m \le h(t)$, WHERE $h(t)$ IS THE CHARACTERISTIC THAT ORIGINATED AT CYLINDER'S OUTER RADIUS AT $t = 0$. $\boxed{\text{AT } t = t \text{ AND } m < h(t)}$, THE CORE FLUID IS ROTATING AT $\hat{V}_0 = \hat{\Omega}_0 \hat{r}$, or IN NONDIM'L FORM,

$$\boxed{V_0 = \frac{\hat{\Omega}_0 \hat{r}}{\Omega_1 \hat{a}} = \left(\frac{\Omega_1 - \Delta\Omega}{\Omega_1}\right) r = (1-\epsilon_1) r} \quad (52)$$

THUS, IN THIS REGION, $\Gamma = (1-\epsilon_1) r^2$ AND THE CHARACTERISTIC EQN (51) IS GIVEN BY

$$(r^2)' = 2\epsilon_0 \left[(1-\epsilon_1) r^2 - r^2\right]$$

INTEGRATE AND USE $r^2(t=0) = r^2(0)$

$$= -2\epsilon_0 \epsilon_1 r^2 = -2c_0 r^2$$

$$\Rightarrow \boxed{r^2(t) = r^2(0) e^{-2c_0 t}} \quad (53) \qquad \boxed{c_0 = \epsilon_0 \epsilon_1}$$

OR $\boxed{m(t) = m(0) e^{-2c_0 t}}$ (54)

$\boxed{m \equiv r^2}$

FROM (54) or (53), WE CAN DETERMINE $h(t)$:

$$h(t) = (1)e^{-2\Omega_0 t}$$

value of $r^2 = 1$ @ CYLINDER'S RADIUS AT $t=0$.

$$\Rightarrow \boxed{h(t) = e^{-2\Omega_0 t}} \qquad (55)$$

$$\boxed{V_\theta(r,t) = \Omega_0 r \qquad r < \sqrt{h(t)} = e^{-\Omega_0 t}}$$

* PHYSICS BEHIND MATH

CORE FLUID AT ANY GIVEN r MAINTAINS A velocity of $\hat{V}_\theta = \hat{\Omega}_0 \hat{r}$ UNTIL 'SPUN-UP' FLUID (WHICH LIES AT $\eta > h(t)$) REACHES r. ONCE THE 'FRONT' OF FASTER MOVING FLUID REACHES r, THEN $\hat{V}_\theta$ AT r ASSUMES SOME TIME DEPENDENT MAGNITUDE.

FOR $\eta = r^2 > h(t)$, THE CHARACTERISTICS ALL ORIGINATE FROM THE CYLINDER'S OUTER RADIUS AND THUS THE VALUE OF Γ ON ANY GIVEN CHARACTERISTIC IN THIS REGION IS EQUAL TO ITS VALUE AT $r=1 \Rightarrow \Gamma = V_0(1,t')(1) = 1$.

HENCE $$\Gamma(r,t) = v_0(r,t)\, r = 1$$

$$\Rightarrow \boxed{v_0(r,t) = 1/r} \qquad (56)$$

WE HAVE TO CAREFUL HERE HOWEVER AND NOTE AS r IS ACTUALLY time-dependent. and is det'd by characteristic eqn:

$$\frac{d(r^2)}{dr} = 2c_0\left[\cancel{r}^{\,1} - r^2\right].$$

$$\Rightarrow (r^2)' + 2c_0 r^2 = 2c_0 \qquad \leftarrow \text{integ. factor} = e^{2c_0t}$$

$$(r^2 e^{2c_0t})' = 2c_0 e^{2c_0t}$$

$$\Rightarrow r^2 = \frac{e^{2c_0t}}{e^{2c_0t}} + A e^{-2c_0t}$$

$$r^2 = 1 + Ae^{-2c_0t}$$

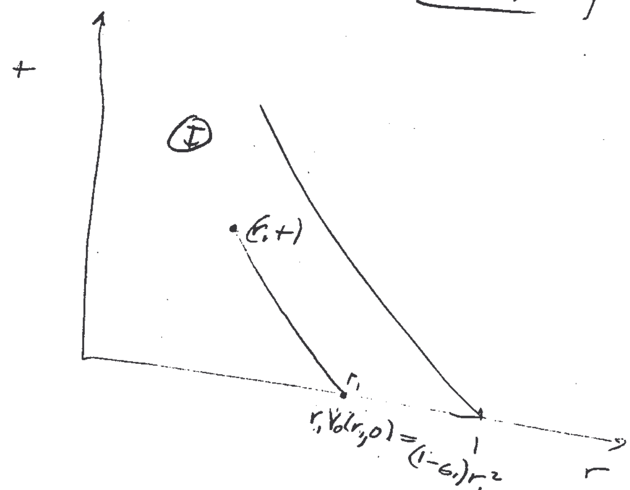
$$r_0^2 = 1 + A \Rightarrow A = (r_0^2 - 1)$$

$$r^2 = 1 + (r_0^2 - 1)e^{-2c_0t} \qquad r_0 \geq 1$$

WE CAN OBTAIN $V_0(r,t)$ IN THE THE REGION $r < h(t)$ AS FOLLOWS, FROM (48c).

$$\frac{dr}{dt} = \underline{u}_1 = \frac{\epsilon_0}{r}\left[rV_0 - r^2 \right] \qquad (56)$$

Let $\boxed{m = r^2}$



or

$$\boxed{\frac{dm}{dt} = 2\epsilon_0\left[rV_0(r,t) - m \right]} \qquad (57)$$

where we choose an arbitrary point (r,t) in region I above, i.e., r and t are treated as constants in the above diff'l. eqn. [We can do this since $rV_0(r,t) = \Gamma = \text{constant} = (1-\epsilon_1)r_1^2$, along the characteristic (curve) shown.]

SEPARATING AND INTEGRATING (57), WE OBTAIN THE FOLLOWING:

$$\int_{m=r^2=r_1^2}^{m=r^2} \frac{dm}{(rV_0 - m)} = 2\epsilon_0 \int_0^t dt$$

$$\Rightarrow \quad \ln\left[\frac{(rV_0 - r^2)}{rV_0 - r_1^2}\right] = -2\epsilon_0 t \qquad (58)$$

THE DENOMINATOR ON THE L.H.S. ABOVE CAN BE WRITTEN AS

$$-r_1^2 + rV_0(r,t) = -r_1^2 + r_1 V_0(r_1, t=0) = r_1^2(1-\epsilon_1) - r_1^2 = -\epsilon_1 r_1^2$$

$$(58) \Rightarrow \quad rV_0(r,t) = r^2 - \epsilon_1 r_1^2 e^{-2\epsilon_0 t}$$

$$\Rightarrow \quad \boxed{V_0(r,t) = r - \epsilon_1 \frac{r_1^2}{r} e^{-2\epsilon_0 t}} \qquad (59)$$

NOW, WE CAN ELIMINATE r_1 FROM (59) BY NOTING THAT ALONG THE CHARACTERISTIC CONNECTING $(r_1, t=0)$ to (r,t), Γ IS CONSTANT.

THUS, ON THE CHARACTERISTIC

$$\Gamma = r_1^2(1-\epsilon_1) = rV_0(r,t) \qquad (60)$$

(INITIAL VALUE AT t=0)

SOLVING (60) FOR r_1^2:

$$r_1^2 = \frac{r V_0(r,t)}{(1-\epsilon_1)} \qquad (61)$$

NOW USING (61) IN (59) WE GET:

$$V_0(r,t) = r - \frac{\epsilon_1}{\cancel{r}} e^{-2\epsilon_0 t}\left[\frac{\cancel{r} V_0(r,t)}{(1-\epsilon_1)}\right]$$

$$\Rightarrow \quad V_0(r,t)\left(1 + \frac{\epsilon_1 e^{-2\epsilon_0 t}}{(1-\epsilon_1)}\right) = r$$

$$\Rightarrow \quad \boxed{V_0(r,t) = \frac{r}{\left[1 + \frac{\epsilon_1}{1-\epsilon_1} e^{-2\epsilon_0 t}\right]}} \qquad (62)$$

$$\boxed{\text{for} \quad r < h(t) = e^{-2\epsilon_0 t}}$$

NOTE THAT (62) SATISFIES THE INITIAL CONDITION $V_0(r,t=0) = (1-\epsilon_1)r$.

ALSO NOTE THAT THE FLUID W/IN THE CORE (AHEAD OF THE SPUN-UP FLUID OVER $h(t) < r \leq 1$) ACHIEVES THE FINAL VELOCITY r AS $t \to \infty$, AS EXPECTED.

10/20/99 (0)

SPIN-UP

BASED ON EXP. OBSERVATION –

- B.L.'S ON CYLINDER ENDS
- CORE OF FLUID WHICH MAINTAINS INITIAL ROTATION RATE (BUT SHRINKS w/ TIME
- 'OUTER' CORE WHERE FLUID ROT. VEL IS $> \Omega_0$ AND WHICH GROWS ACROSS CYLINDER

B.L. REGION

- GET δ, u_s, w_s SIMULTANEOUSLY BY SCALING: $(\hat{V}_s = \hat{\Omega}_i \hat{a}$
  - BALANCE radial centrifugal force by " viscous force

$$\boxed{\rho \frac{V_i^2}{r} \approx \hat{\rho} \hat{\Omega}_i^2 \hat{a} \approx \mu u_{zz} \approx \hat{\mu} \hat{u}_s / \hat{\delta}^2} \quad (1)$$

$$\Rightarrow \boxed{\hat{u}_s \approx \hat{\rho} \hat{\Omega}_i^2 \hat{a} \, \hat{\delta}^2 / \hat{\mu}} \quad (1)$$

- CONTY: $\boxed{\hat{w}_s \approx \hat{\delta} \hat{u}_s / \hat{a}}$ (2)

- ARGUING THAT B.L. KEPT THIN BY ENTRAINMENT:

  VORTICITY EQN:

  vert convection OF ω_θ into BL $\approx$ outward diffusion OF ω_θ

$$\Downarrow$$

$$w_s \frac{\omega_\theta}{\delta} \approx \nu \frac{\omega_\theta}{\delta^2} \Rightarrow \boxed{w_s \approx \nu / \delta} \quad (3)$$

10/20/94

(2)

$(2)+(3) \Rightarrow \quad \frac{\nu}{\delta} \approx \frac{\delta u_s}{a}$

$\Rightarrow \delta^2 \approx \nu a / u_s \qquad (4)$

$(1)+(4) \Rightarrow \quad u_s^2 \approx \frac{\cancel{\rho}\Omega_1^2 a \,\cancel{\nu} a}{\cancel{\mu}}$

$\Rightarrow \boxed{\hat{u}_s \approx \hat{\Omega}_1 \hat{a}} \qquad (5)$

$(4) \Rightarrow \boxed{\delta \approx \sqrt{\frac{\nu a}{\Omega_1 a}} = \sqrt{\frac{\nu}{\Omega_1}}} \qquad (6)$

$(6)+(3) \Rightarrow \boxed{w_s \approx \frac{\nu}{\delta} \approx \nu / \sqrt{\nu/\Omega_1} \approx \sqrt{\nu \Omega_1}} \qquad (7)$

NOTE $\boxed{V_s \approx \Omega_1 a} \qquad (8)$

CORE SCALES

$\boxed{V_s \approx \hat{\Omega}_1 \hat{a}} \qquad (9)$

GOT U_s BY BALANCING RADIALLY OUTWARD FLOW FROM B.L. BY RADIALLY INWARD " INTO CORE

$u_s 2\cancel{\pi} \cancel{a} \delta \approx U_s 2\cancel{\pi} a L$

$(1) \Rightarrow \boxed{U_s \approx \frac{\delta}{L} u_s = \epsilon \Omega_1 a} \qquad \epsilon$ = EKMAN NO. $= \delta/L$

10/20/49 (3)

$\underline{w_s}$ ⇒ get by 1) equating to w_s in B.L. or 2) by cont'y w/in core

1) $\boxed{\underline{w_s} \approx w_s = \sqrt{\nu \Omega_1}}$ (11)

2) $w_s \approx \frac{u_s}{a} L$

$\approx \frac{u_s L}{a} = \epsilon \Omega_1 a \frac{L}{\cancel{a}}$

$= \frac{\delta}{\cancel{L}} \Omega_1 \cancel{L} = \sqrt{\frac{\nu}{\Omega_1}} \Omega_1$

$\boxed{\underline{w_s} = \sqrt{\nu \Omega_1}}$ (11)

$\underline{t_s} = t_{\text{adjustment}} \approx t_a \approx \frac{\text{vol of cyl}}{Q \text{ through B.L.}} \approx \frac{a^2 L}{Q}$

$Q \approx u_s \delta a \approx \Omega_1 a^2 \delta$

$\Rightarrow \boxed{t_a \approx \frac{\cancel{a^2} L}{\Omega_1 \cancel{a^2} \delta} = \Omega_1^{-1} \epsilon^{-1}}$ (12)

NOTE $\boxed{t_{BL} = \Omega_1^{-1}}$ (13)

⇒ since $t_{BL} \ll t_a$ the boundary layer is quasi stationary — our non-dim'l eqns should reflect this (i.e., $\frac{\partial}{\partial t}$ term should be small)

10/20/09 (2)

NON DIM'LIZE -

2 APPROACHES

I) 1) SCALE velocities, lengths, times as found above

e.g. in B-L

$$u = \frac{\hat{u}}{\hat{u}_s} = \frac{\hat{u}}{\Omega_1 a}$$

$$v = \frac{\hat{v}}{\hat{v}_s} = \frac{\hat{v}}{\Omega_1 \hat{a}}$$

$$w = \frac{\hat{w}}{\hat{w}_s} = \frac{\hat{w}}{\sqrt{\nu \Omega_1}}$$

$$(r, z, t) = \left(\frac{\hat{r}}{a}, \frac{\hat{z}}{\delta}, \frac{\hat{t}}{\hat{t}_s}\right)$$

2) NON-DIM'LIZE GOV. EQNS

3) ASSUME SOL'N OF FORM

$$\boxed{\begin{aligned} u &= u_0 + \epsilon u_1 + \epsilon^2 u_2 \\ v &= v_0 + \epsilon v_1 + \epsilon^2 v_2 \\ w &= w_0 + \epsilon w_1 + \epsilon^2 w_2 \end{aligned}} \qquad (14)$$

WHERE $u_0, v_0, w_0, u_1, v_1, w_1$ are all fns. of pos. & time $O(1)$

* IN THIS APPROACH, SMALL COEFFICIENTS MULTIPLY SUBDOMINANT FORCE TERMS IN MOM. EQNS SO THAT WHEN (14) IS INSERTED INTO DIM'LESS EQNS, THOSE SMALL TERMS DROP OUT.

12/20/99 (5)

· APPROACH II

1) USE SINGLE (USUALLY LARGEST) SCALE TO NON-DIM'LIZE VEL'S, LENGTHS, TIMES. (THIS IS APPROACH IN NOTES)

2) NON-DIM'LIZE GOV. EQNS — NOTE THAT SUBDOMINANT TERMS DO NOT MULTIPLY SMALL COEFFICIENTS (IN EQUIV. EQNS). IN ORDER TO GET CORRECT DIMENSIONLESS SCALES ON SUBDOMINANT TERMS, ASSUME SOLNS. WHICH CONTAIN CORRECT ACCT FOR SCALES:

3) ASSUME ASYMPTOTIC SOLNS OR FORM

E.G. $\boxed{\text{IN B.C.}}$

$$\boxed{\begin{aligned} u &= u_0 + \epsilon u_1 + \ldots \\ v &= v_0 + \epsilon v_1 + \ldots \\ w &= 0 + \epsilon w_1 \\ p &= p_0 + \epsilon p_1 \end{aligned}}$$

(NOTE $\hat{P}_S = \hat{\rho}\hat{V}^2 \approx \hat{\rho}\hat{\mathcal{R}}_1\hat{a}^2$ IN BOTH B.C. & CORE)

$\boxed{\text{IN CORE}}$

$$\boxed{\begin{aligned} \underline{u} &= 0 + \epsilon \underline{u}_1 \\ \underline{V} &= \underline{V_0} + \epsilon \underline{V}_1 \\ \underline{W} &= 0 + \epsilon \underline{W}_1 \\ \underline{P} &= P_0 + \epsilon P_1 \end{aligned}}$$

& (16) IN NOTES

10/27/99 (B)

DIM'L EQNS $\overset{B.C.}{\Rightarrow}$ (9)

NONDIM'LIZED EQNS/BC $\Rightarrow$ (10a) – (10h)

ATTACK CORE FIRST

ASS. (16) FOR CORE SOLN, NN SOLN.

$\Rightarrow$ O(1):

$\underline{\underline{r}}$: $\frac{\underline{V_0}^2}{r} = \underline{P_{0,r}}$

(physics – at leading order radial P gradient in core determined by centrifugal fcn

$\Rightarrow$ HAVE TO GO TO 1ST ORDER TO DETERMINE CORE FLOW $\Rightarrow$ CONSISTENT W/ SCALING AND PHYSICS

$\underline{\underline{\theta}}$: NO O(1) terms

$\underline{\underline{z}}$: $\underline{P_{0,z}} = 0$

CONT'Y no O(1) terms

$P_{0,z} = 0 \Rightarrow \frac{\partial}{\partial r}(\underline{\underline{r}}) \Rightarrow \underline{V_{0,z}} = 0$

$\underline{V_0} = \underline{V_0}(r,t)$

O(E) : (14a) – (14d)

$(r\underline{V_0})_{,t} + u_1 (r\underline{V_0})_{,r} = 0$

(A) θ: $\boxed{\Gamma_{,t} + u_1 \Gamma_{,r} = 0}$ $\boxed{\Gamma = r\underline{V_0}(r,t)}$

will use this to solve for $\underline{V_0}$ $\Rightarrow$ WILL NEED u_1 FIRST THOUGH

10/20/99 ②

$O(\epsilon)$ cont'd

$\underline{\underline{r}}$: $\dfrac{-2U_0V_1}{r} = -P_{1,r}$

$\underline{\underline{z}}$: $P_{1,z} = 0$

CONT'Y $(ru_1)_{,r} + (rw_1)_{,z} = 0$ (C)

NOTE THAT $\dfrac{\partial(A)}{\partial z} \Rightarrow u_{1,z} = 0$

$\Rightarrow \boxed{u_1 = u_1(r,t)}$ (B)

~~* IMPORTANT POINT — TO SOLVE ~~EDDY~~ FLOW, MUST SOLVE $O(\epsilon)$ PROBLEM IN CORE~~

~~RESCALE B.C.'~~

REEXPRESS DIM'LESS GOV'ING EQNS / B.C.' (10a)–(10h)

IN FORM APPROPRIATE FOR B.C.'S

HERE, ~~THE~~ $\hat{r}_s$ REMAINS $= \hat{a}$, BUT $\hat{z}_s = \hat{\delta}$

$\hat{V}_s = \hat{\Omega}_1\hat{a}$, $\hat{u}_s = \hat{\Omega}_1\hat{a}$, $\hat{w}_s = \sqrt{\nu\Omega_1} = \sqrt{\dfrac{\nu}{\Omega_1}}\,\Omega_1 = \hat{\delta}\hat{\Omega}_1$

$= \epsilon\hat{\Omega}_1\hat{L} = \epsilon\Omega_1\hat{a}\dfrac{\hat{L}}{a} = \dfrac{\epsilon}{\epsilon_0}\hat{V}_s$

$= O(\epsilon\hat{V}_s)$

~~NOTE~~ RATHER THAN STARTING W/ DIM'L EQNS AND ROTATING DERIVATION OF (10a–h), WE CAN SIMPLY RESCALE DIM'LESS Z, ~~ADJUST~~ KEEP THE UNIFORM VEL. SCALE, AND AS IN CORE, ACCT FOR ORDER OF MAG. DIFF'S

10/20/99
(8)

in vel. comps ~~[illegible]~~ in our ASS'D

SLN.

$\Rightarrow$ rescaled B.L. eqns given by
(23a)–(23d). pg. (13)

<u>NOW ASS'D SOLN. IN B.L.</u>

$$\boxed{\begin{aligned} u &= u_0 + \epsilon u_1 \\ v &= v_0 + \epsilon v_1 \\ w &= 0 + \epsilon w_1 \\ p &= p_0 + \epsilon p_1 \end{aligned}} \quad (D)$$

PLUG D INTO (23);

$O(1)$: (26a)–(26d): pg (13)

From ~~[illegible]~~ $p_0 \neq fn(z) \Rightarrow p_0$ w/in
B.L. (@ any r) $\equiv \underline{P_0}$ outside B.L.
From $O(1)$ r-mom. eqn in core

$$p_{0,r} = \underline{P_{0,r}} = \underline{\underline{V_0}}^2 / r$$

$\Rightarrow$ r-mom in B.L. becomes

$$\boxed{u_0 u_{0,r} + w_1 u_{0,z} - \frac{v_0^2}{r} = -\frac{\underline{\underline{V_0}}^2}{r} + u_{0,zz}}$$

10/20/94 (a)

for $\frac{\hat{\Omega}_1 - \hat{\Omega}_0}{\hat{\Omega}_0} \ll \cancel{1}$ CAN OBTAIN CORRECTED SOLN

$$\hat{\Omega}_1 = \hat{\Omega}_0 + \frac{\Delta\hat{\Omega}}{\hat{\Omega}_1} =$$

BY ASSUMING

$$B.C \quad (u_0, v_0, w_0) = (0, r, 0) + \underbrace{\frac{\Delta\hat{\Omega}}{\hat{\Omega}_1}}_{\epsilon_1}(\tilde{u}, \tilde{v}, \tilde{w})$$

$$\text{CON} \quad (\_ \_ \_) = (0, r, 0) + \epsilon_1(\tilde{u}, \tilde{v}, 0)$$

$\Rightarrow$ WE ARE THUS MAKING AN $O(\epsilon_1)$ ADJUSTMENT TO $O(1)$ soln. (u_0, v_0, w_0) and $(\underline{u}_0, \underline{v}_0, \underline{w}_0) \Rightarrow \epsilon_1 \gg \epsilon$

$$\boxed{r \text{ mp} \quad \epsilon_1^2 \tilde{u}\tilde{u}_r + \epsilon_1 w_1 \tilde{u}_z + \left(\frac{\overbrace{v_0^2}^{r^2+\epsilon_1^2\tilde{v}^2+2r\epsilon_1\tilde{v}} - \underbrace{\underline{v}_0^2}_{r^2+\epsilon_1\tilde{v}^2+2r\epsilon_1\tilde{v}}}{r}\right) = \epsilon_1 \tilde{u}_{zz}} \quad *$$

re. w_1, since we assumed that the $\underline{\text{adjustment}}$ to $O(1)$ soln (u_0, v_0, w_0) was of the form $\epsilon_1(\tilde{u}, \tilde{v}, \tilde{w})$, THEN WE CAN ALSO assume that $\underline{\text{adjustment}}$ to $O(\epsilon)$ soln is of form

$$(u_1, v_1, w_1) = \underbrace{(0, 0, 0)}_{\substack{\text{First order} \\ \text{soln during} \\ \text{solid bdy rotation}}} + \epsilon(\tilde{u}_1, \tilde{v}_1, \tilde{w}_1)$$

$$\Rightarrow \boxed{w_1 = O(\epsilon_1)}$$

10/20/99 ①

TMUS

(XII) Ⓐ Blooming

$$\boxed{2(\underline{\tilde{V}} - \tilde{V}) = \tilde{u}_{,zz}} \qquad (A1)$$

(264) $\Rightarrow$

$$u_1\left(v_{0,r} + \frac{v_0}{r}\right) + \epsilon_0\, w_1 v_{0,z} = v_{0,zz}$$

$$\Rightarrow \boxed{2\tilde{u} = \tilde{v}_{,zz}} \qquad (A2)$$

Let $\quad \psi = \underline{\tilde{v}}(r,t) - \tilde{v}(r,z,t)$

$$\Rightarrow \tilde{v}_{,zz} = -\psi_{,zz}$$

$$(A1) \rightarrow \boxed{2\psi = \tilde{u}_{,zz}} \qquad (A3)$$

$$(A2) \Rightarrow \boxed{-2\tilde{u} = \psi_{,zz}} \qquad (A4)$$

$\times i$

def

$$(A3) + (A4) \Rightarrow \boxed{2(\psi - i\tilde{u}) = (\tilde{u} + i\psi)_{,zz}} \qquad (A5)$$

Let $\quad \phi = \tilde{u} + i\psi$

$$(A5) \Rightarrow \boxed{-2i\phi = \phi_{,zz}}$$

$$\Rightarrow \phi_{,zz} + 2i\phi = 0$$

$$\boxed{\phi = c_1 e^{mz} + c_2 e^{-mz}}$$

$m^2 = -2i$

$m = (2i)^{1/2}$

10/20/84 (11)

B.C.'s on ϕ

$$\phi = \tilde{u} + i\psi$$

$$\phi(r, \underline{z}=0, t) = 0 + i(\tilde{\underline{V}}(r,t) - 0)$$

$$= i\underline{V}(r,t)$$

$$\phi(r, \underline{z}\to\infty, t) = 0 + i(\tilde{\underline{V}}(r,t) - \underline{V}(r,t))$$

$$= 0$$

---

soln given (40) & (41).

NOW HAVE $(u_0, v_0, w_0) = (0, r, 0) + \epsilon_1(\bar{u}, \bar{v}, \bar{w})$

NEED CONT SOLN;

$$\Gamma_{1t} + \underline{u}_1\Gamma_{1r} = 0$$

$\Rightarrow$ ~~NEED~~ u_1

SHOWED $u_1 \neq fn(\underline{z})$

$$\underline{z} = \epsilon z \qquad z_{BL} = \frac{\underline{z}}{\delta} =$$

$$\int_0^{1/\bar{a}} \bar{u}\, dz = 0 = \epsilon \int_0^\infty u\, d\underline{z} + \int_0^{1/\epsilon_0} \underline{u}\, dz = 0$$

$$= \epsilon \int_0^\infty \epsilon_1 \tilde{u}\, d\underline{z} + \int_0^{1/\epsilon_0} (\underbrace{u_0}_{0} + \epsilon\underline{u}_1)\, dz = 0$$