LINEAR STABILITY CONT'D —

RAYLEIGH-BENARD CONVECTION

INTRODUCTION —

THIS PROBLEM WAS EXPERIMENTALLY INVESTIGATED BY BENARD (1900) AND WAS FIRST REPORTED BY THOMPSON (1882). HERE, A THIN INITIALLY STAGNANT FLUID LAYER IS HEATED FROM BELOW. AT SOME CRITICAL VALUE OF THE TEMPERATURE GRADIENT ACROSS THE LAYER, THE STAGNANT FLUID SUDDENLY STARTS MOVING. PHYSICALLY, MOTION IS INITIATED WHEN THE STABILIZING INFLUENCES OF THERMAL CONDUCTION AND VISCOSITY ARE OVERCOME BY THE DESTABILIZING EFFECT OF BUOYANCY. [CONDUCTION DIFFUSES HEAT AWAY FROM WARM, RELATIVELY BUOYANT LOW-LYING FLUID TO COOLER OVERLYING FLUID, SUPPRESSING THE WARM FLUID'S TENDENCY TO RISE INTO THE COOL FLUID. LIKEWISE, VISCOSITY TENDS TO ACT AGAINST UPWARDLY ACTING BUOYANT FORCES. WHEN THE TEMP. GRADIENT IS LARGE ENOUGH, DENSITY DIFFERENCES BETWEEN THE WARM LOW-LYING FLUID AND THE COOL

OVER-LYING FLUID BECOME LARGE ENOUGH THAT BUOYANCY CAN OVERCOME FRICTION (VISCOSITY) AND THE DIFFUSION OF HEAT BY CONDUCTION.] THE FLOW APPEARS AS A HIGHLY REGULAR COLLECTION OF POLYGONAL CELLS (SEE ATTACHED PHOTO), WHERE EACH CELL TYPICALLY HAS 4 TO 7 SIDES. HERE, WE WILL FOLLOW THE SAME BASIC PROCEDURE ILLUSTRATED IN THE FIRST TWO EXAMPLES IN ORDER TO ANALYZE HOW THE BASIC (STAGNANT) FLOW GIVES WAY TO CELLULAR, BUOYANCY-DRIVEN CONVECTION. WE WILL FIND THAT AT A CRITICAL VALUE OF THE RAYLEIGH NUMBER, Ra (DEFINED BELOW), THE STAGNANT BASIC STATE GIVES WAY TO A STEADY AND STABLE BUOYANCY-DRIVEN FLOW, i.e., WE'LL FIND THAT AN EXCHANGE OF STABILITIES OCCURS (WHERE $\omega = 0$ AT MARGINAL STABILITY, $\sigma = 0$).

ANALYSIS

WE WILL START WITH THE DIMENSIONLESS GOVERNING EQNS. DERIVED IN THE ATTACHMENT. IMPORTANT POINTS TO NOTE ARE AS FOLLOWS:

1) THE BOUSSINESQ APPROXIMATION IN WHICH DENSITY IS ASSUMED CONSTANT EVERYWHERE EXCEPT IN THE BODY FORCE TERM, $\rho\vec{g}$, IS INVOKED, 2) AS SHOWN IN THE ATTACHMENT, SINCE $\alpha \equiv \frac{1}{\rho_0}\frac{\partial\rho}{\partial\theta}\Big|_{\theta_0}$ IS SMALL FOR LIQUIDS AND GASES. AND SINCE $\alpha > \frac{1}{\mu_0}\frac{d\mu}{d\theta}, \frac{1}{k_0}\frac{dk}{d\theta}, \frac{1}{c_0}\frac{dc}{d\theta}$, THEN SINCE WE NEGLECT TERMS OF ORDER α AND SMALLER, THEN μ, k, c CAN BE TAKEN AS CONSTANT. (WHERE μ, k, AND c ARE THE FLUID VISCOSITY, THERMAL CONDUCTIVITY AND SPECIFIC HEAT), 3) THE CONT'Y EQN. ASSUMES THE FORM $\nabla \cdot \underline{V} = 0$. THIS FOLLOWS FROM THE FOLLOWING SCALING ARGUMENT:

EXACT CONTIN EQN IS GIVEN BY

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \underline{V} = 0$$

SINCE $\rho = \rho_0 + \left.\frac{\partial \rho}{\partial \theta}\right|_{\theta_0}(\theta - \theta_0)$ (TAYLOR EXPANSION ABOUT $\theta = \theta_0$)

$$= \rho_0(1 - \alpha(\theta - \theta_0)) \quad \left(\alpha \equiv -\frac{1}{\rho_0}\left.\frac{\partial \rho}{\partial \theta}\right|_{\theta_0}\right)$$

THEN CONTIN EQN CAN BE WRITTEN AS

$$-\alpha \cancel{\rho_0} \frac{D\theta}{Dt} + \cancel{\rho_0}(1 - \alpha(\theta - \theta_0))\nabla \cdot \underline{V} = 0$$

OR

$$\frac{-\alpha}{(1 - \alpha\Delta\theta)}\frac{D\theta}{Dt} + \nabla \cdot \underline{V} = 0$$

WHERE $\Delta\theta = \theta - \theta_0$

$$\Downarrow \qquad\qquad \Downarrow$$

$$O\left(\alpha \Delta\theta \frac{w}{d}\right) \qquad O\left(\frac{w}{d}\right)$$

WHERE w = vertical velocity component AND d = thickness of fluid layer

$$\Downarrow$$

$$O\left(\frac{1}{\rho_0}\frac{\Delta\rho}{\cancel{\Delta\theta}}\cancel{\Delta\theta}\frac{w}{d}\right)$$

THUS,

$$\frac{\frac{-\alpha}{(1-\alpha\Delta\theta)}\frac{D\theta}{Dt}}{\nabla \cdot \underline{V}} = O\left(\frac{\Delta\rho}{\rho_0}\right)$$

Since $\frac{\Delta\rho}{\rho_0} \ll 1$ THEN CONTIN CAN BE APPROXIMATED AS

$$\boxed{\nabla \cdot \underline{V} = 0}\left(+ O\left(\frac{\Delta\rho}{\rho_0}\right)\right) \qquad (2a)$$

4) SIMILAR SCALING ARGUMENTS CAN BE USED TO OBTAIN THE DIMENSIONAL GOVERNING EQNS GIVEN BY (7.9) AND (7.10) IN ATTACHMENT:

(7.9) $$\boxed{\frac{Du_i}{Dt} = -\frac{\partial}{\partial x_i}\left(\frac{p}{\rho_0} + gz\right) - \alpha g(\theta_0 - \theta)\delta_{i3} + \nu\nabla^2 u_i}$$

(7.10) $$\boxed{\frac{D\theta}{Dt} = \bar{K}\nabla^2\theta} \qquad \bar{K} = \frac{k}{\rho c}$$

NOTE, THESE GOVERN BASIC STATE AND PERTURBED FLOW.

$\boxed{\text{BASIC STATE}}$ - PRIOR TO ONSET OF BUOYANT CONVECTION, THE FLUID IS STAGNANT. (USE * TO DENOTE DIM'L QTYS).

THUS, $$\boxed{u_i^* = 0} \qquad i = 1, 2, 3 \qquad (1)$$

FROM (7.10), WHERE $\theta^*_{,t} = 0$ (AND THUS $\frac{D\theta^*}{Dt} = 0$),

$$\nabla^2\theta^* = 0 \qquad (2)$$

THE BOTTOM BOUNDARY, $z^* = 0$, IS AT $\theta^* = \theta_0^*$. WHILE THE UPPER BNDRY, $z^* = d^*$, IS AT $\theta^* = \theta_1^*$. SINCE LAYER IS EFFECTIVELY INFINITE IN x & y DIRECTIONS AND SINCE THERE ARE NO INTERNAL HEAT SOURCES W/IN FLUID LAYER

$$\theta^* \neq fn(x^*, y^*)$$

$$\Rightarrow \quad \nabla^2 \theta^* = \boxed{\frac{d^2\theta^*}{dz^{*2}} = 0}$$

$$\Rightarrow \quad \theta^* = c_1 z^* + c_2$$

B.C. @ $z^*=0 \Rightarrow \quad \theta_0^* = c_1(0) + c_2 \Rightarrow c_2 = \theta_0^*$

B.C. @ $z^*=d^* \Rightarrow \quad \theta_1^* = c_1 d^* + \theta_0^* \rightarrow c_1 = \dfrac{\theta_1^* - \theta_0^*}{d^*}$

USING BOOK'S NOTATION

$$c_1 = -\beta = \frac{\theta_1^* - \theta_0^*}{d^*}$$

$$\Rightarrow \boxed{\theta^*(z^*) = \theta_0^* - \beta z^*} \qquad (3)$$

THE PRESSURE DISTRIBUTION IN THE BASIC STATE FOLLOWS FROM (7.9).

$$x\text{-DIR}: \quad \frac{\partial p^*}{\partial x^*} = 0 \Rightarrow p^* \neq fn(x^*)$$

$$y\text{-DIR} \quad \frac{\partial p^*}{\partial y^*} = 0 \Rightarrow p^* \neq fn(y^*)$$

$$z\text{-DIR} \quad 0 = -\frac{1}{\rho_0^*}\frac{dp^*}{dz^*} - g^* - \alpha g(\theta_0^* - \theta^*) \quad (4)$$

NOTE, THIS (AND EQN. (7.9)) CAN BE EASILY DERIVED STARTING FROM BASIC MOMENTUM EQN., W/ ρ IN THE BODY FORCE TERM REPLACED

BY $\rho_0(1-\alpha(\theta-\theta_0))$;

$$\rho_0^* \frac{Du_i^*}{Dt^*} = -P_{,i}^* + \mu \nabla^2 u_i^* - \rho_0^*(1-\alpha(\theta^*-\theta_0^*))g^* \delta_{i3}$$

$$\Rightarrow \quad \frac{Du_i^*}{Dt^*} = -\frac{\partial}{\partial x_i}\left(\frac{P^*}{\rho_0}\right) - \frac{\partial}{\partial x_i^*}(g^* z^*) + \alpha(\theta^*-\theta_0^*)g^*\delta_{i3} + \nu \nabla^2 u_i^*$$

$\Rightarrow$ THIS IS EQN. (7.9)

RETURNING TO Z-MOMENTUM EQN (4) AND USING (3) FOR θ^* WE OBTAIN

$$\frac{dP^*}{dz^*} = -\rho_0^* g^* + \alpha g^*(-\beta z^*)\rho_0^*$$

$$\Rightarrow \quad P^* = -\rho_0^* g^*\left(z^* + \alpha\beta \frac{z^{*2}}{2}\right) + C_1$$

@ $z^*=0$ $\quad P^* = P_0^* = C_1$

THUS,

$$\boxed{P^*(z^*) = P_0^* - \rho_0^* g^*\left(z^* + \alpha\beta\frac{z^{*2}}{2}\right)} \quad (5)$$

LINEARIZATION

WE NOW LINEARIZE THE DIM'L CONT'Y (2A), MOMENTUM (7.9) AND ENERGY EQNS. (7.10).

LET VELOCITY, PRESSURE AND TEMPERATURE IN DISTURBED FLOW BE EXPRESSED AS

$$\underline{u}^* = \underline{u}^{*'}$$
$$p^* = p_s^*(z) + p^{*'}(\underline{x}', t) \qquad (6)$$
$$\theta^* = \theta_s^*(z) + \theta^{*'}(\underline{x}^{*'}, t)$$

WHERE

SUBSCRIPT "S" DENOTES BASIC STATE
$$\begin{cases} p_s^*(z) = p_0^* - \rho_0^* g^*\left(z^* + \frac{a\beta}{2} z^{*2}\right) & \text{(eqn 5)} \\ \theta_s^*(z) = \theta_0^* - \beta z^* & \text{(eqn 3)} \end{cases}$$

AND WHERE $p^{*'}$, $\underline{u}^{*'}$, AND $\theta^{*'}$ ARE THE TIME AND SPACE-DEPENDENT PERTURBATIONS TO THE BASIC STATE.

INSERTING (6) INTO (2a), (2.9) & (2.10) AND NEGLECTING PRODUCTS OF PERTURBATION QTYS., WE OBTAIN THE LINEARIZED EQNS.:

(2a) ⇒ $$\boxed{\nabla \cdot \underline{u}^{*'} = 0} \qquad (7)$$

(2.9) ⇒
$$\underline{u}^{*'}_{,t} = -\frac{1}{\rho_0^*}\left(\overset{(\mathrm{I})}{\frac{\partial p_s^*}{\partial z^*}\hat{e}_3} + \nabla p^{*'}\right) - \overset{(\mathrm{II})}{g^*\hat{e}_3}$$
$$+ \overset{(\mathrm{II}')}{a g^*(\theta_s^* - \theta_0^*)\hat{e}_3} + a g^* \theta^{*'} \hat{e}_3 + \nu \nabla^2 \underline{u}^{*'} \qquad (8)$$

IN THE BASIC STATE, TERMS (I), (II) AND (III) IN EQN. (8) SUM TO 0;

SEE EQN. (7.9) WHERE $\underline{u}^* = \underline{u}^*_s = 0$.

THUS, EQN. (8) SIMPLIFIES TO

$$\boxed{\underline{u}^{*\prime}_{,t} = -\frac{1}{\rho_0^*}\nabla p^{*\prime} + \alpha g \theta^{*\prime}\hat{e}_3 + \nu \nabla^2 \underline{u}^{*\prime}} \quad (9)$$

NEXT (7.10) $\Rightarrow$ $\theta^{*\prime}_{,t} + w^{*\prime}\theta^*_{s,z} = \mathbb{K}\left[\theta^*_{s,zz} + \nabla^2\theta^{*\prime}\right]$

HOWEVER, SINCE $\theta^*_{s,zz} = 0$ WHILE $\theta^*_{s,z} = -\beta$, THEN LINEARIZED ENERGY EQN BECOMES:

$$\boxed{\theta^{*\prime}_{,t} - w^{*\prime}\beta = \mathbb{K}\nabla^2\theta^{*\prime}} \quad (10)$$

NONDIMENSIONALIZATION –

- EXPECT VELOCITY AND TEMPS. TO VARY MOST SIGNIFICANTLY OVER DISTANCES ON THE ORDER OF THE LAYER THICKNESS, d^* THUS LENGTH SCALE $= d^*$
- THE TEMP. SCALE IS $\theta_0^* - \theta_1^* = \beta d^*$
- THE VELOCITY SCALE U_s^* IS DETERMINED BY BALANCING VERTICAL CONVECTION, $w^*\theta^*_{,z}$, AGAINST THE DOMINANT CONDUCTION TERM, $\mathbb{K}^*\theta^*_{,zz}$:

$$w^*\theta^*_{/z^*} \approx \overline{K}^*\theta^*_{/zz}$$

$$\Rightarrow \frac{U_s^*}{d^*} = \frac{\overline{K}^*}{(d^*)^2} \Rightarrow \boxed{U_s^* = \frac{\overline{K}^*}{d^*}}$$

NOTE THAT ANOTHER CHOICE FOR THE VELOCITY SCALE MIGHT BE OBTAINED BY BALANCING AXIAL ADVECTION, $w^*w^*_{/z^*}$, AGAINST BUOYANCY, $g^*\alpha^*(\theta_0^*-\theta_1^*) = g^*\alpha^*\beta d^*$. THIS CHOICE PRESUMES THAT ADVECTION IS IMPORTANT, WHICH MAY NOT BE TRUE FOR SLIGHTLY SUPER-CRITICAL FLOW (i.e., FLOW AT RAYLEIGH NOS. SLIGHTLY LARGER THAN THE Ra WHERE FLOW BEGINS). THE FIRST CHOICE FOR U_s^* ASSUMES THAT ONCE FLOW BEGINS, AXIAL CONVECTIVE HEAT TRANSFER IS COMPARABLE TO AXIAL CONDUCTION.

IF WE USE TYPICAL VALUES OF

$\overline{K}^* \approx 10^{-6}\,m^2/s$, $d^* = 10^{-2}\,m$, $\theta_0^*-\theta_1^* = 10K$, $\alpha^* = 5(10^{-4})\,K^{-1}$, WE FIND THAT $U_s^* = \frac{\overline{K}^*}{d^*} = 10^{-4}\,m/s$ WHILE $U_s^* = \sqrt{g^*\alpha^*\beta^* d^{*2}} \simeq 2.2(10^{-2})\,m/s$. THUS, FOR SLIGHTLY SUPER-CRITICAL FLOW $U_s^* = \frac{\overline{K}^*}{d^*}$ WOULD APPEAR TO BE MORE APPROPRIATE.

⊕ THE PRESSURE SCALE IS DETERMINED BY EITHER INERTIA $\left(\frac{\overline{\rho}^* U_s^{*2}}{d^*}\right)$ OR VISCOSITY $\left(\mu^* U_s^*/d^{*2}\right)$. ALTHOUGH IT IS READILY SHOWN THAT

VISCOUS TERM IS LARGER THAN THE INERTIA TERM, WE'LL FOLLOW THE BOOK (AND TRADITION) AND USE $\rho U_s^{*2}/d^*$ AS THE PRESSURE SCALE.

⊕ <u>NOTE</u> – SINCE WE ARE NOT PURSUING AN ASYMPTOTIC SOLN., THE ACTUAL SCALES USED ARE NOT CRITICAL TO THE ANALYSIS. CHOOSING OTHER SCALES WOULD LEAD TO AN EIGENVALUE RELATION FOR GROWTH RATE S STATED IN TERMS OF NONDIMENSIONAL PARAMETERS THAT DIFFER FROM THOSE IN THE ATTACHMENT. THE STABILITY CHARACTERISTICS, HOWEVER, (AS ENCOMPASSED BY THE EIGENVALUE RELATION FOR S) WOULD NOT BE AFFECTED. THE REASON FOR SCALE-INSENSITIVITY HERE IS THAT THE PROBLEM (GIVEN BY (7), (9), (10)) IS <u>LINEAR</u>. NOTE – NONDIMENSIONALIZATION NOT NECESSARY IN LINEAR STAB. USED HERE LARGELY AS A CONVENIENCE TO INTERPRETING STABILITY CHARACTERISTICS.

<u>DIMENSIONLESS</u> <u>EQNS</u>:

USING THE ABOVE SCALES IN (7), (9) AND (10) LEADS TO THE FOLLOWING DIMENSIONLESS EQUIVALENTS:

(11) $$\nabla \cdot \underline{u} = 0$$

(12) $$\underline{u}_{,t} = -\nabla P + R Pr\, \theta \hat{e}_3 + Pr \nabla^2 \underline{u}$$

(13) $$\theta_{,t} - w = \nabla^2 \theta$$

STRATEGY: 1) USE (11)-(13) TO OBTAIN SINGLE EQN. IN W, 2) DERIVE B.C.'S ON W AT $z = 0, 1$, 3) GUESS/PROPOSE A FORM FOR THE NORMAL MODES, 4) OBTAIN EIGENVALUE RELATIONSHIP FOR S, 5) ANALYZE STABILITY BASED ON S.

TAKE CURL OF (12):

$$\underline{\omega}_{,t} = R Pr\, \nabla \times \theta \hat{e}_3 + Pr \nabla^2 \underline{\omega} \qquad (14)$$

(NOTE, SINCE ∇^2 AND $\nabla \times$ ARE LINEAR OPERATORS, WE CAN INTERCHANGE THEIR ORDER. THUS, FOR EXAMPLE $\nabla \times (\nabla^2 \underline{u}) = \nabla^2 (\nabla \times \underline{u}) = \nabla^2 \underline{\omega}$. THIS CAN BE EASILY VERIFIED USING INDEX NOTATION)

NOW TAKE CURL OF (14):

(15) $$\nabla \times (\nabla \times \underline{u})_{,t} = R Pr\, \nabla \times \nabla \times (\theta \hat{e}_3) + Pr \nabla^2 (\nabla \times \nabla \times \underline{u})$$

USE INDEX NOTATION TO PROVE OR RECALL THAT

$$\nabla\times\nabla\times\underline{u} = -\nabla^2\underline{u} + \nabla(\nabla\cdot\underline{u})$$
$$= -\nabla^2\underline{u} \quad \text{SINCE} \quad \nabla\cdot\underline{u}=0$$

IN ADDITION,

$$(\nabla\times\nabla\times\theta\hat{e}_3)_i = \varepsilon_{ijk}\, b_{k,j}$$

$$b_k = (\nabla\times\theta\hat{e}_3)_k$$
$$= \varepsilon_{klm}\, C_{m,l}$$
$$C_m = \theta\delta_{3m}$$

$$= \varepsilon_{ijk}\varepsilon_{klm}(\theta\delta_{3m})_{,lj}$$
$$= \varepsilon_{kij}\varepsilon_{klm}(\theta\delta_{3m})_{,lj}$$
$$= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\,\theta_{,lj}\,\delta_{3m}$$
$$= \theta_{,im}\delta_{3m} - \theta_{,ll}\,\delta_{3i}$$
$$= \theta_{,i3} - \nabla^2\theta\,\delta_{3i}$$

$$\Rightarrow \nabla\times\nabla\times\theta\hat{e}_3 = \frac{\partial}{\partial z}(\nabla\theta) - \nabla^2\theta\,\hat{e}_3$$

THUS, EQN. (15) BECOMES:

$$(16) \quad \boxed{\nabla^2\underline{u}_{,t} = RPr\left[\nabla^2\theta\,\hat{e}_3 - \frac{\partial}{\partial z}\nabla\theta\right] + Pr\,\nabla^2(\nabla^2\underline{u})}$$

THE z-COMP. OF (16) IS;

$$\boxed{\nabla^2 w_{,t} = R P_r \left[\nabla_1^2 \theta\right] + P_r \nabla^2(\nabla^2 w)} \quad (17)$$

WHERE

$$\nabla_1^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

(13): $\theta_{,t} - w = \nabla^2 \theta$

or $\left(\frac{\partial}{\partial t} - \nabla^2\right)\theta = w$ (18)

BY INSPECTION OF (17) & (18), WE CAN ELIMINATE θ FROM (17) BY OPERATING ON (18) w/ $R P_r \nabla_1^2$ AND ON (17) w/ $\left(\frac{\partial}{\partial t} - \nabla^2\right)$:

$$\left(\frac{\partial}{\partial t} - \nabla^2\right)(17) \Rightarrow \left(\frac{\partial}{\partial t} - \nabla^2\right)\nabla^2 w_{,t} = R P_r \left(\frac{\partial}{\partial t} - \nabla^2\right)\nabla_1^2 \theta + P_r \left(\frac{\partial}{\partial t} - \nabla^2\right)\nabla^2(\nabla^2 w) \quad (19)$$

$$R P_r \nabla_1^2 (18) \Rightarrow R P_r \nabla_1^2 \left(\frac{\partial}{\partial t} - \nabla^2\right)\theta = R P_r \left(\frac{\partial}{\partial t} - \nabla^2\right)\nabla_1^2 \theta = R P_r \nabla_1^2 w \quad (20)$$

THUS USING R.H.S. OF (20) TO REPLACE TERM INVOLVING θ IN (19), WE OBTAIN

$$\left(\frac{\partial}{\partial t}-\nabla^2\right)\nabla^2 w_{/t} = RP_r \nabla_1^2 w + P_r\left(\frac{\partial}{\partial t}-\nabla^2\right)\nabla^2\left(\nabla^2 w\right) \quad (21)$$

OR

$$\left(\frac{\partial}{\partial t}-\nabla^2\right)\left(\frac{\partial}{\partial t}-P_r\nabla^2\right)\nabla^2 w = RP_r \nabla_1^2 w$$

OR

$$\boxed{\left(\frac{\partial}{\partial t}-\nabla^2\right)\left(\frac{1}{P_r}\frac{\partial}{\partial t}-\nabla^2\right)\nabla^2 w = R\nabla_1^2 w} \quad (22)$$

(THIS IS (8.14) IN ATTACHMENT)
(22) WILL LEAD TO THE EIGENVALUE RELATION FOR S.

<u>BOUNDARY CONDITIONS ON W</u>

ALTHOUGH IT IS POSSIBLE TO DERIVE B.C.'S FOR A FLUID LAYER CONFINED BETWEEN RIGID BOUNDARIES, OR BETWEEN A RIGID LOWER BNDRY AND A FREE UPPER BNDRY, WE WILL <u>ILLUSTRATE</u> THE SOLN. APPROACH USING THE PHYSICALLY UNREALISTIC CASE OF A LAYER LYING BETWEEN TWO <u>FREE</u> BNDRIES. THIS IS THE ONLY NON-NUMERICAL CASE WHICH ALLOWS EVALUATION OF THE EIGENVALUE RELN. FOR S.

INSPECTING (22) WE SEE THAT THIS LINEAR P.D.E. IS OF 6th ORDER (HIGHEST ORDER DERIVATIVES ARE 6th ORDER IN SPACE). THUS, WE'LL NEED 6 B.C.'S IN THE Z-DIRECTION (3 ON EACH BNDRY $Z=0,1$). ALTHOUGH (22) ALSO HAS 6th ORDER DERIVATIVES W/ RESPECT TO X AND Y, WE WILL LOOK FOR <u>NORMAL MODES</u> THAT ARE PERIODIC IN X AND Y, CONSISTENT W/ EXPERIMENTALLY <u>OBSERVED</u> PERIODIC CONVECTION CELLS

AT EITHER FREE BOUNDARY, THE CHANGES IN THE NORMAL AND TANGENTIAL STRESS (FROM BASIC STATE VALUES) ARE TAKEN TO BE ZERO. PHYSICALLY, THIS MEANS THAT WE ASSUME

AMBIENT PRESSURE ACTING ON EACH FREE BNDRY: IS UNAFFECTED BY PERTURBATIONS TO THE BASIC STATE (+ CHANGES IN FREE SURFACE CURVATURE DUE TO PERTURBATIONS ARE ALSO NEGLIGIBLE) AND THAT EXTERNAL SHEAR STRESSES (IF PRESENT) ARE LIKEWISE UNAFFECTED.

AS DETAILED IN THE ATTACHMENT (PP. 40–41), THE NORMAL STRESS CONDITION, $-P_i^* = \sigma_{ij} n_i n_j$, ONCE NON-DIMENSIONALIZED, LEADS TO THE CONDITION THAT BOTH FREE SURFACES ARE FLAT [WHERE THE ERROR IN THIS CONDITION IS ON THE ORDER OF α, AS SHOWN IN (8.24)]. THUS

$$z = 1 + F_u(x,y,t) = 1 \quad \text{ON UPPER FREE SURFACE}$$

$$z = 0 + F_\ell(x,y,t) = 0 \quad \text{ON LOWER FREE SURF.}$$

WHERE F_u AND F_ℓ ARE (DIM'LESS) PERTURBATIONS TO THE FREE SURFACE. IN OTHER WORDS, $\boxed{F_u = F_\ell = 0}$

THE DIM'L KINEMATIC CONDITION IS GIVEN BY

$$\frac{dz^*}{dt^*} = \frac{dF^*}{dt^*} \Rightarrow w^* = F^*_{,x} u^* + F^*_{,y} v^* + F^*_{,t^*}$$

LINEARIZING USING $F^* = F_o^* + F^{*\prime}$ AND $\underline{u} = \underline{u}'$ (WHERE $F_o^* = d^*$ @ $z^* = d^*$ AND $F_o^* = 0$ @ $z^* = 0$), WE OBTAIN

$$w^{*\prime} = F^{*\prime}_{,t}$$

HOWEVER, SINCE $F^{*'} = F_u d^* = 0$ @ $z = 1$
AND $F^{*'} = F_l d^* = 0$ @ $z = 0$, THEN

$$w^{*'} = 0 \quad @ \quad z^* = 0, d^*$$

OR EQUIVALENTLY (IN NONDIM'L FORM):

$$\boxed{w = 0 \quad @ \quad z = 0, 1} \qquad (23)$$

NOW SINCE TANGENTIAL STRESSES ARE ZERO, THEN

$$\sigma^*_{ij} n_j t_i = 0 = -p\cancel{\delta_{ij} n_j t_i}^{\,0} + \mu^*(u^*_{i,j} + u^*_{j,i}) n_j t_i$$

where t_i = ith comp. of unit tangent vector

CONSIDERING THE SURFACE AT $z^* = d^*$, $\underline{n} = (0,0,1) = \hat{e}_3$. RECALLING THAT THE ith COMPONENT OF THE ith COMPONENT OF THE LOCAL STRESS VECTOR IS GIVEN BY $t_i^{(\hat{n})} = \sigma_{ij} n_j$, WE CAN IDENTIFY $\mu^*(u^*_{i,j} + u^*_{j,i}) n_j$ AS ith COMPONENT OF THE 'VISCOUS' STRESS VECTOR. THUS, THE EQN.

$$[\mu^*(u^{*'}_{i,j} + u^{*'}_{j,i}) n_j] t_i = 0 \qquad (24)$$

STATES THAT THE TANGENTIAL PROJECTION OR COMPONENT OF THE VISCOUS STRESS VECTOR ON THE

FREE SURFACE IS ZERO, i.e., THAT BOTH THE X AND Y COMPONENTS OF THE VISCOUS STRESS VECTOR ARE ZERO. THUS, SINCE $\underline{n} = \hat{e}_3$, WE HAVE 2 EQNS;

X-COMP. $(i=1, j=3$ in (24)):

$$u^{*\prime}_{1,3} + u^{*\prime}_{3,1} = 0 \qquad (25)$$

Y-COMP $(i=2, j=3$ in (24)):

$$u^{*\prime}_{2,3} + u^{*\prime}_{3,2} = 0 \qquad (26)$$

IN NONDIMENSIONAL FORM, THESE BECOME:

$$\boxed{\begin{aligned} u_{,z} + w_{,x} &= 0 \\ v_{,z} + w_{,y} &= 0 \end{aligned}}_{\;z=0,1} \qquad \begin{aligned}(27)\\(28)\end{aligned}$$

SINCE $w = 0$ ON EITHER SURFACE, (27) AND (28) YIELD

$$z=0,1 \quad \boxed{\begin{aligned} u_{,z} &= 0 \\ v_{,z} &= 0 \end{aligned}} \qquad (29)$$

NOW USING CONT'Y ON EITHER SURFACE AND DIFFERENTIATING W.R.T. z GIVES

$$(u_{1x} + v_{1y} + w_{1z})_{,z} = 0 \quad \text{ON} \quad z = 0, 1$$

$$\Rightarrow \cancel{(u_{1z})_{,x}}^{\,0} + \cancel{(v_{1z})_{,x}}^{\,0} + w_{1zz} = 0 \qquad ''$$

$$\Rightarrow \quad \boxed{w_{1zz} = 0 \quad \text{ON} \quad z = 0, 1} \quad (30)$$

(23) AND (30) PROVIDE 4 B.C.'S IN Z-DIRECTION. THE REMAINING 2 WILL BE OBTAINED ONCE NORMAL MODES ARE INTRODUCED.

NORMAL MODES

FOLLOWING RAYLEIGH, WE TAKE NORMAL MODES OF THE FORM

$$w = W(z) f(x,y) e^{st} \qquad (31)$$

$$\theta = T(z) f(x,y) e^{st} \qquad (32)$$

THESE MODES HAVE THE SAME HORIZONTAL FORM ON EVERY PLANE Z. THEY ARE SUGGESTED AND ARE CONSISTENT WITH THE EXPERIMENTALLY OBSERVED CONVECTION CELLS DESCRIBED ABOVE (AND ILLUSTRATED IN THE ATTACHMENT).

INSERT (31) AND (32) INTO (13) TO OBTAIN:

$$sTf - Wf = fT'' + T\nabla_1^2 f$$

OR

$$\frac{sT - W - T''}{T} = \frac{\nabla_1^2 f}{f} \qquad (33)$$

SINCE R.H.S. OF THIS EQN IS A fn. OF X AND y WHILE THE LEFT IS A FN. OF z, THEN BOTH SIDES MUST EQUAL A CONSTANT, $-a^2$.
THUS (33) LEADS TO

$$sT - W - T'' + a^2 T = 0 \qquad (34)$$

$$\nabla_1^2 f + a^2 f = 0 \qquad (35)$$

[WE CHOOSE A NEGATIVE SEPN. CONSTANT IN ORDER TO OBTAIN A WAVE-LIKE EQN. FOR $f(x,y)$ IN (35).]

IN ORDER TO OBTAIN THE LAST TWO B.C.'S ON W, WE INSERT THE NORMAL MODE SOLNS. IN (31) AND (32) INTO EQN (17):

$$s\left(W\overset{-a^2 f}{\cancel{\nabla_1^2 f}} + fW''\right) = RP_r T\overset{-a^2 f}{\cancel{\nabla_1^2 f}} + P_r \nabla^2\left(W\overset{-a^2 f}{\cancel{\nabla_1^2 f}} + fW''\right)$$

$$\Rightarrow f\left[s(-a^2 W + W'') + RP_r a^2 T\right] = P_r\left[-a^2\left(W\overset{-a^2 f}{\cancel{\nabla_1^2 f}} + fW''\right) + W''\underset{-a^2 f}{\cancel{\nabla_1^2 f}} + fW''''\right]$$

$$\Rightarrow \quad \boxed{(W'' - a^2 W)(s + a^2 P_r) - P_r(W'''' - a^2 W'') = -R P_r a^2 T}$$

$$\Rightarrow \quad (s + a^2 P_r - P_r D^2)(D^2 - a^2) W = -R P_r a^2 T$$

WHERE $D^2 = \frac{d^2}{dz^2}$

$$\Rightarrow \quad \boxed{(D^2 - a^2)(D^2 - a^2 - s/P_r) W = R a^2 T} \quad (36)$$

THIS IS EQN 8.38 IN ATTACHMENT.

NOW APPLYING CONDITIONS THAT

$W = W(z) f(x,y) e^{st} = 0$, $W_{,zz} = W''(z) f(x,y) e^{st} = 0$

AND $\theta = T(z) f(x,y) e^{st} = 0$ (WHERE TEMP. B.C. FOLLOWS BY ASSUMING THAT UPPER AND LOWER BNDRYS REMAIN AT FIXED TEMPS.), IT FOLLOWS THAT

$$(37) \quad \boxed{W(z) = W''(z) = T(z) = 0 \quad \text{ON } z = 0, 1}$$

THUS, USING THESE CONDITIONS IN (36) (WHICH APPLIES AT ALL POINTS, INCLUDING $z = 0, 1$), WE ARRIVE AT THE FINAL TWO B.C.'S ON $W(z)$:

$$\boxed{D^4W = 0} \quad \text{ON } z = 0, 1 \quad (38)$$

THE GOVERNING EQN FOR $W(z)$ FOLLOWS BY INSERTING $w = W(z)\, f(x,y)\, e^{st}$ INTO (22); THE RESULT IS OBTAINED USING THE SAME APPROACH USED TO DERIVE (36):

$$\boxed{(D^2 - a^2)(D^2 - a^2 - s)(D^2 - a^2 - s/P_r)W = -a^2 R W} \quad (39)$$

THE B.C.'S ON $W(z)$ CAN BE COLLECTED:

$$(40) \quad \boxed{W = D^2W = D^4W = 0 \quad \text{ON } z = 0, 1}$$

FROM (39) IT IS SEEN THAT W APPEARS AS D^6W, D^4W, D^2W, AND W.

THE B.C.'S ON W IN (40) SUGGEST AN INFINITE NUMBER OF SOLNS OF THE FORM $\boxed{W_n = \sin(n\pi z)}$ WHERE $n = 1, 2, 3, \ldots$

INDEED, USING THIS SOLN IN (39) LEADS TO AN EIGENVALUE RELATIONSHIP FOR s. PHYSICALLY, A SINUSOIDAL VARIATION FOR THE VERTICAL DISTURBANCE VELOCITY SEEMS INTUITIVELY REASONABLE GIVEN THE SYMMETRY OF THE B.C.'S ON EACH SURFACE.

USING $W = \sin(n\pi z)$ IN (39) YIELDS:

(41) $$\boxed{\left(-(n\pi)^2 - a^2\right)\left(-(n\pi)^2 - a^2 - s\right)\left(-(n\pi)^2 - a^2 - s/Pr\right) = -a^2 R}$$

YOU SHOULD VERIFY (BY INSPECTION OR BY CALCULATION) THAT EACH APPLICATION OF THE DIFFERENTIAL OPERATOR $D^2 = \frac{d^2}{dz^2}$ YIELDS $-(n\pi)^2$. BUT $(n\pi)^2 + a^2 = b^2$ SO THAT (41) BECOMES

$$(b^2+s)\left(b^2+\frac{s}{Pr}\right) - \frac{a^2R}{b^2} = 0$$

$$\Rightarrow \quad \frac{s^2}{Pr} + s(b^2)\left(1+\frac{1}{Pr}\right) + \left(b^4 - \frac{a^2R}{b^2}\right) = 0$$

$$s^2 + s b^2 (Pr+1) + Pr\left(b^4 - \frac{a^2R}{b^2}\right) = 0$$

(42) $$\Rightarrow \quad \boxed{\begin{gathered} s = \frac{-b^2(Pr+1)}{2} \pm \left[\frac{1}{4}b^4(Pr+1)^2 - Pr\left(b^4 - \frac{a^2R}{b^2}\right)\right]^{1/2} \\ b^2 = (n\pi)^2 + a^2 \end{gathered}}$$

SINCE $\frac{1}{4}b^4(Pr+1)^2 - Pr\, b^4 = \frac{1}{4}b^4(Pr-1)^2$ THEN (42) CAN BE WRITTEN AS

(42) $$\boxed{\begin{gathered} s = \frac{-b^2(Pr+1)}{2} \pm \left[\frac{1}{4}b^4(Pr-1)^2 + \frac{a^2 Pr R}{b^2}\right]^{1/2} \\ b^2 = (n\pi)^2 + a^2 \end{gathered}}$$

THIS IS (10.5) IN ATTACHMENT.

⊕ INSPECTING (42), WE SEE THAT $\sigma = Re(s)$ IS ALWAYS NEGATIVE AND THE LAYER IS STABLE WHEN $R < 0$. SINCE $R = -\frac{g\alpha d^4}{K\nu}\frac{d\theta^*}{dz^*}$, THEN $R < 0$ corresponds to $\frac{d\theta^*}{dz^*} > 0$, i.e., FLUID TEMP. INCREASING OR EQUIVALENTLY, DENSITY DECREASING, W/ INCREASING z^*. THIS IS CONSISTENT W/ THE PHYSICAL FACT THAT A STABLY STRATIFIED FLUID LAYER (i.e., $\frac{d\rho^*}{dz^*} < 0$) IS STABLE.

⊕ LOOKING AT (42), WE SEE THAT FOR ANY GIVEN WAVE NUMBER a, PRANDTL NUMBER, Pr, AND MODE NUMBER n, THERE'S A POSITIVE VALUE OF R, SAY $R_{cn} (= fn(a, n, Pr))$, AT WHICH THE SQUARE ROOT TERM BECOMES EQUAL TO $\frac{b^2}{2}(Pr+1)$.

FOR $R > R_{cn}$, $\sigma > 0$ AND THE MODE BECOMES UNSTABLE. FOR $R < R_{cn}$, $\sigma < 0$ AND THE MODE IS STABLE. THUS, SINCE $\sigma = 0$ AT R_{cn} AND SINCE THE BASIC STATE BECOMES UNSTABLE FOR VALUES OF R SLIGHTLY LARGER THAN R_{cn}, THE SYSTEM IS MARGINALLY STABLE AT R_{cn}. IN ADDITION, since R_{cn} depends on the mode number n (for given Pr and a), AND SINCE THERE IS A COUNTABLE INFINITY OF n's, THERE IS A COUNTABLE INFINITY OF R_{cn}'s.

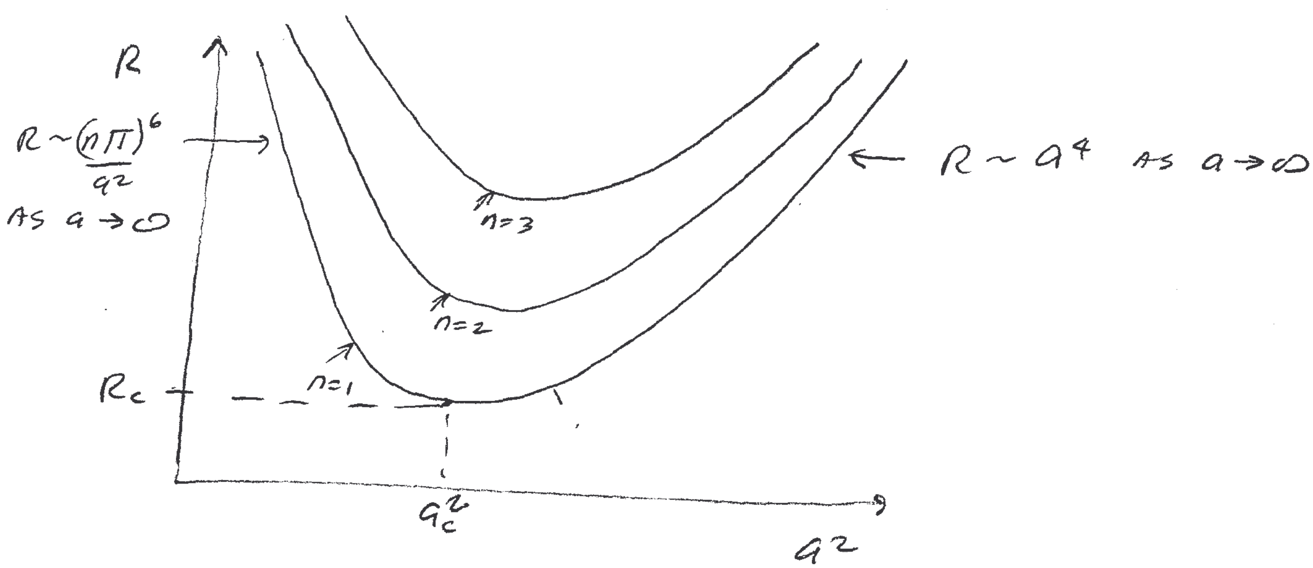
INTUITIVELY, WE EXPECT THAT THE SMALLEST R_{cn} WILL BE THE CRITICAL RAYLEIGH NUMBER, R_c, WHERE R_c IS DEFINED BY THE CONDITIONS THAT THE BASIC STATE IS STABLE FOR ALL $R < R_c$ AND IS UNSTABLE FOR ANY $R > R_c$. INDEED, AS PROVEN IN SECTION 9.2 IN THE ATTACHMENT, THIS INTUITIVE IDEA IS CORRECT.

(4) WHEN $R = R_c$ (OR $R = R_{cn}$ FOR THAT MATTER), THE MODE IS MARGINALLY STABLE, AS DISCUSSED ABOVE. SINCE $\omega \equiv Im(s) = 0$ AT MARGINAL STABILITY, THEN EXCHANGE OF STABILITIES OCCURS, i.e., THE INSTABILITY SETS IN AS A STABLE SECONDARY FLOW. THIS IS CONSISTENT W/ EXPERIMENTAL OBSERVATIONS SHOWING THAT THE THERMAL INSTABILITY SETTLES INTO STABLE CELLULAR FLOW.

- IN ORDER TO DETERMINE R_c, i.e., THE SMALLEST VALUE OF R AT MARGINAL STABILITY, WE SET $s = 0$ $(\equiv \sigma + i\omega)$ IN EQ. (41) (or (42)). WE CAN DO THIS SINCE WE KNOW (BASED ON PRECEDING DISCUSSION) THAT $\omega = Im(s) = 0$ AT MARGINAL STABILITY. THUS, FROM (41) WE GET

$$R = \left[(n\pi)^2 + a^2\right]^3 / a^2 = fn(n,a) \qquad (43)$$

FROM (43) IT IS CLEAR THAT AT ANY GIVEN WAVE NUMBER a, THE SMALLEST R CORRESPONDS TO THE $n=1$ mode. THIS CAN BE SEEN GRAPHICALLY BY PLOTTING R VS a^2 AND n:



FROM GRAPH IT IS CLEAR THAT THE SMALLEST VALUE OF R OCCURS ON THE $n=1$ CURVE. IN ORDER TO DETERMINE R_c AND a_c, FIND MINIMUM OF

$$R = \frac{(\pi^2 + a^2)^3}{a^2}$$

$$\frac{dR}{da^2} = \frac{3(\pi^2+a^2)^2}{a^2} - \frac{(\pi^2+a^2)^3}{a^4} = 0$$

$$\Rightarrow 3a^2 = \pi^2 + a^2$$

$$\Rightarrow \boxed{a = a_c = \pi/\sqrt{2}} \qquad (44)$$

THUS $\boxed{R_c = \frac{\pi^6(1+\frac{1}{2})^3}{(\pi^2/2)}} = \pi^4\left(\frac{27}{4}\right) = 657.5$

(44) MEANS THAT FOR R SLIGHTLY GREATER THAN R_c, THE THE THERMAL INSTABILITY SETS IN WITH A HORIZONTAL WAVELENGTH

$$\lambda_c^* = \frac{2\pi}{a_c^*} = \frac{2\pi}{a_c/d^*} = \frac{2^{3/2}\pi d^*}{\pi} = 2^{3/2} d^* = \boxed{2.83 d^*}$$