2/2/00
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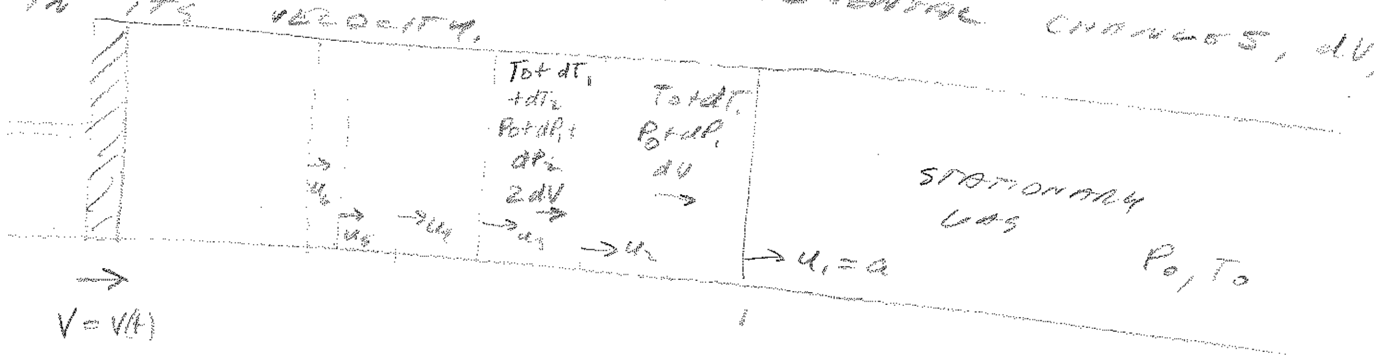
LECTURE III
NORMAL SHOCKS; OBLIQUE SHOCKS

(I) NORMAL SHOCKS

A) SHOCK FORMATION: WE CAN GET AN INTUITIVE UNDERSTANDING OF HOW NORMAL SHOCKS FORM BY CONSIDERING THE RAPID ACCELERATION OF A PISTON INTO AN AN AIR-FILLED TUBE. TO VISUALIZE THE PROCESS, IMAGINE THAT THE PISTON ACCELERATES FROM 0 TO V THROUGH A SERIES OF INCREMENTAL CHANGES, dV, IN ITS VELOCITY.



WAVE ① FORMED WHEN PISTON ACCELERATES FROM 0 TO dV; ITS SPEED IS $a = \sqrt{\gamma R T_0}$, WHERE T_0 IS THE INITIAL GAS TEMP. THE VELOCITY OF THE GAS BEHIND WAVE ① AND IN FRONT OF WAVE ② IS dV. IN ADDITION, DUE TO A SMALL COMPRESS, THE PRESSURE IS P_0+dP. WAVE ② FORMS WHEN THE PISTON VELOCITY JUMPS FROM dV TO $2dV$. THE SPEED OF WAVE ② <u>RELATIVE</u> TO THE MOVING GAS BEHIND WAVE ① IS $a + da$, WHERE da IS > 0 SINCE $T = T_0 + dT_1$.

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THUS THE <u>ABSOLUTE</u> SPEED OF WAVE (2) (RELATIVE TO THE FIXED DUCT WALLS) IS: $dV + a + da$. AS BEFORE, THE SPEED OF THE GAS BEHIND WAVE (2) IS $2dV$; THE PRESSURE AND TEMPERATURE LIKEWISE INCREASE TO $P_0 + dP_1 + dP_2$ AND $T_0 + dT_1 + dT_2$.

THUS, SINCE WAVE (2) IS MOVING FASTER THAN WAVE (1), IT EVENTUALLY OVERTAKES WAVE (1). WHEN THIS HAPPENS, THE PRESSURE AND TEMPERATURE JUMPS ACROSS THE <u>LEADING</u> COALESCED WAVE INCREASE FROM dP_1 TO $dP_1 + dP_2$ AND FROM dT_1 TO $dT_1 + dT_2$. BY SIMILAR REASONING, WE CAN SEE THAT EACH INCREMENTAL INCREASE IN THE PISTON'S VELOCITY PRODUCES A WAVE WHICH MOVES FASTER THAN ALL PRECEDING WAVES. ALTHOUGH THE SPEED OF THE COALESCED WAVE CONTINUES TO INCREASE AS MORE TRAILING WAVES CATCH UP TO IT, ITS ABSOLUTE SPEED IS NEVER GREATER THAN THOSE OF THE TRAILING (INFINITESIMAL) WAVES BEHIND IT. (PROOF LATER.) THE COALESCED WAVE THUS CONTINUES TO INCREASE IN STRENGTH (AS MEASURED BY THE

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SIZE OF THE PRESSURE JUMP ACROSS IT) EVENTUALLY FORMING A (MOVING) <u>NORMAL</u> <u>SHOCK</u>.

B) $\boxed{\text{STATIONARY NORMAL SHOCK RELATIONS}}$

WE'LL FIRST DERIVE GOVERNING EQNS. FOR A NORMAL SHOCK THAT REMAINS IN A <u>FIXED</u> <u>POSITION</u> <u>RELATIVE</u> <u>TO</u> A <u>NON-MOVING</u> <u>REFERENCE</u> <u>FRAME</u>. THE RESULTS CAN THEN BE EASILY ADAPTED TO THE CASE OF A <u>MOVING</u> NORMAL SHOCK - WE'LL SIMPLY FIX A COORDINATE SYSTEM TO THE MOVING SHOCK AND CAREFULLY BOOK KEEP THE UPSTREAM AND DOWNSTREAM GAS SPEEDS ACROSS THE SHOCK. LATER, THESE SAME RELATIONS (FOR STATIONARY NORMAL SHOCKS) WILL BE ADAPTED TO PROBLEMS INVOLVING OBLIQUE SHOCKS.

– $\boxed{\text{PHYSICAL FEATURES OF NORMAL SHOCK FLOW}}$

$M_2 < 1$	
$u_2 < u_1$	$\leftarrow u_1$ $M_1 > 1$
$P_2 > P_1$	P_1
$T_2 > T_1$	T_1
$\rho_2 > \rho_1$	ρ_1
$S_2 > S_1$	S_1

$T_{02} = T_{01}$

$P_{02} < P_{01}$

IN NORMAL SHOCK FLOW, THE VELOCITY UPSTREAM & DOWNSTREAM OF SHOCK IS PERPENDICULAR TO THE SHOCK.

ASSUMPTIONS

1) CALORICALLY PERFECT GAS ⇒

C_p = CONST

C_v = "

$e = C_v T$

$h = C_p T$

$P = \rho R T$

2) STEADY STATE CONDITIONS

3) UNIFORM PROPERTIES UPSTREAM & DOWNSTREAM OF SHOCK

4) NO DISSOCIATION OR IONIZATION ACROSS SHOCK

OBJECTIVE DERIVE WORKING RELATIONSHIPS GIVING

$\frac{P_2}{P_1}$, $\frac{\rho_2}{\rho_1}$, $\frac{T_2}{T_1}$, $\frac{P_{02}}{P_{01}}$, AND M_2 EACH AS A FN. OF THE UPSTREAM MACH NO. M_1.

APPROACH MANIPULATE CONT'Y, CONS. OF MASS AND CONS. OF ENERGY EQNS. INTO FORMS INVOLVING $\frac{\rho_2}{\rho_1}$, $\frac{a_2}{a_1}$ and $fn(M_1, M_2)$; ELIMINATE $\frac{\rho_2}{\rho_1}$, $\frac{a_2}{a_1}$, and SOLVE FOR M_2 IN TERMS OF M_1; BACK OUT DESIRED EQNS.

CONS. OF MASS (ACROSS SHOCK):

$$\rho_1 u_1 = \rho_2 u_2$$

DIVIDE BY a_1:

$$\rho_1 \frac{u_1}{a_1} = \rho_2 \frac{u_2}{a_1} = \rho_2 \frac{u_2}{a_2} \frac{a_2}{a_1} = \boxed{\rho_2 M_2 \left(\frac{a_2}{a_1}\right) = \rho_1 M_1}$$

$$\Rightarrow \boxed{\frac{\rho_2}{\rho_1} = \left(\frac{M_1}{M_2}\right)\left(\frac{a_1}{a_2}\right)} \qquad (1)$$

CONS. OF MOMENTUM:

$$p_1 + \rho_1 u_1^2 = p_2 + \rho_2 u_2^2 \qquad (2)$$

$\Rightarrow$ BUT $p = \rho R T$

$$p = \frac{\rho}{\gamma} a^2 \qquad (3)$$

$(3)+(2) \Rightarrow$

$$\frac{\rho_1 a_1^2}{\gamma} + \rho_1 u_1^2 = \frac{\rho_2 a_2^2}{\gamma} + \rho_2 u_2^2 \qquad (4)$$

$(4)/\frac{a_1^2}{\gamma} \Rightarrow$

$$\rho_1 + \gamma \rho_1 M_1^2 = \rho_2 \left(\frac{a_2}{a_1}\right)^2 + \gamma \rho_2 M_2^2 \left(\frac{a_2}{a_1}\right)^2 \qquad (5)$$

$$\Rightarrow \rho_1 (1 + \gamma M_1^2) = \rho_2 \left(\frac{a_2}{a_1}\right)^2 (1 + \gamma M_2^2)$$

$$\Rightarrow \boxed{\frac{\rho_2}{\rho_1} = \left(\frac{a_1}{a_2}\right)^2 \left(\frac{1+\gamma M_1^2}{1+\gamma M_2^2}\right)} \qquad (6)$$

CONS. OF ENERGY:

$$h_1 + \frac{1}{2} u_1^2 = h_2 + \frac{1}{2} u_2^2$$

$$C_p T_1 + \frac{1}{2} u_1^2 = C_p T_2 + \frac{1}{2} u_2^2$$

$$\frac{\cancel{\gamma R T_1}^{\,a_1^2}}{\gamma - 1} + \frac{1}{2} u_1^2 = \frac{\cancel{\gamma R T_2}^{\,a_2^2}}{\gamma - 1} + \frac{1}{2} u_2^2$$

DIVIDE EACH EQN BY a_1^2 ;

$$\frac{1}{\gamma - 1} + \frac{1}{2} M_1^2 = \left(\frac{a_2}{a_1}\right)^2 \left(\frac{1}{\gamma - 1}\right) + \frac{1}{2} M_2^2 \left(\frac{a_2}{a_1}\right)^2 \qquad (7)$$

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$$\Rightarrow \boxed{\left(\frac{a_2}{a_1}\right)^2 = \frac{\left(1 + \left(\frac{\gamma-1}{2}\right)M_1^2\right)}{\left(1 + \left(\frac{\gamma-1}{2}\right)M_2^2\right)}} \qquad (8)$$

NOW eliminate $\left(\frac{P_2}{P_1}\right)$ FROM (1) & (6);

(1) & (6) $\Rightarrow$

$$\left(\frac{M_1}{M_2}\right)\left(\frac{a_1}{a_2}\right) = \left(\frac{a_1}{a_2}\right)^2 \left(\frac{1+\gamma M_1^2}{1+\gamma M_2^2}\right) \qquad (9)$$

$$(9) \Rightarrow \frac{a_1}{a_2} = \left(\frac{M_1}{M_2}\right)\left(\frac{1+\gamma M_2^2}{1+\gamma M_1^2}\right) \qquad (10)$$

SQUARE (10) AND ELIMINATE $\left(\frac{a_1}{a_2}\right)$ FROM THE RESULTING EQN AND (8);

$$\left(\frac{M_1}{M_2}\right)^2 \left(\frac{1+\gamma M_2^2}{1+\gamma M_1^2}\right)^2 = \frac{2+(\gamma-1)M_2^2}{2+(\gamma-1)M_1^2} \qquad (11)$$

FOLLOWING SOME ALGEBRA (SEE ATTACHMENT), THIS CAN BE SOLVED FOR M_2^2 IN TERMS OF M_1^2;

$$\boxed{M_2^2 = \frac{\left[M_1^2 + \left(\frac{2}{\gamma-1}\right)\right]}{\left[\left(\frac{2\gamma}{\gamma-1}\right)M_1^2 - 1\right]}} \qquad (12)$$

PROOF THAT (12) FOLLOWS FROM (11):

FROM (11):

$$(2M_1^2 + (\gamma-1)M_1^4)(1 + 2\gamma M_2^2 + \gamma^2 M_2^4)$$

$$(2M_2^2 + (\gamma-1)M_2^4)(1 + 2\gamma M_1^2 + \gamma^2 M_1^4)$$

$$\Rightarrow \quad 2(M_1^2 - M_2^2) + (\gamma-1)(M_1^4 - M_2^4) + 2\gamma(\gamma-1)(M_2^2 M_1^4 - M_1^2 M_2^4) = 0$$

or $$2(M_1^2 - M_2^2) + (\gamma-1)(M_1^2 + M_2^2)(M_1^2 - M_2^2) + (\gamma-1)(2\gamma)M_1^2 M_2^2(M_1^2 - M_2^2) = 0$$

or since $M_1 \neq M_2$

$$2 + (\gamma-1)(M_1^2 + M_2^2) + (\gamma-1)2\gamma M_1^2 M_2^2 = 0$$

$$\Rightarrow \quad M_2^2\left[(\gamma-1) + (\gamma-1)2\gamma M_1^2\right] = (1-\gamma)M_1^2 - 2$$

$$\Rightarrow \quad M_2^2 = \frac{M_1^2 - \left(\frac{2}{1-\gamma}\right)}{1 + 2\gamma M_1^2} = \frac{(1-\gamma)M_1^2 - 2}{(1-\gamma) + 2\gamma M_1^2}$$

$$\boxed{M_2^2 = \frac{(\gamma-1)M_1^2 + 2}{2\gamma M_1^2 - (\gamma-1)}}$$

END OF PROOF

NOW USE (12) IN (8) TO GET T_2/T_1:

$$(8) + (12) \rightarrow \left(\frac{a_2^2}{a_1^2}\right) = \boxed{\frac{T_2}{T_1} = \frac{1+\left(\frac{\gamma-1}{2}\right)M_1^2}{1+\left(\frac{\gamma-1}{2}\right)f(M_1^2)}} \qquad (13)$$

WHERE

$$\boxed{M_2^2 = f(M_1^2) = \frac{(\gamma-1)M_1^2+2}{2\gamma M_1^2-(\gamma-1)}} \qquad (14)$$

TO GET $\frac{\rho_2}{\rho_1}$, FIRST USE IDEAL GAS LAW TO GET:

$$\frac{P_2}{P_1} = \frac{\rho_2 R T_2}{\rho_1 R T_1} = \left(\frac{\rho_2}{\rho_1}\right)\left(\frac{a_2^2}{a_1^2}\right) \qquad (15)$$

NOW USE (6) TO ELIMINATE $\frac{\rho_2}{\rho_1}$ FROM (15):

$$\boxed{\frac{P_2}{P_1} = \frac{1+\gamma M_1^2}{1+\gamma M_2^2}}$$

OR

$$\boxed{\frac{P_2}{P_1} = \frac{1+\gamma M_1^2}{1+\gamma f(M_1^2)}} \qquad (16)$$

THIS LAST EQN CAN BE MANIPULATED TO GIVE

$$\boxed{\frac{P_2}{P_1} = \frac{2\gamma M_1^2-(\gamma-1)}{\gamma+1}} \qquad (16a)$$

(NO PROOF)

Given $\frac{P_2}{P_1}$ (in eqn. (16)) and $\frac{T_2}{T_1}$ (eqn. (13)), we can now get $\frac{\rho_2}{\rho_1}$:

$$\boxed{\frac{\rho_2}{\rho_1} = \frac{(P_2/RT_2)}{(P_1/RT_1)} = \underbrace{\left(\frac{P_2}{P_1}\right)}_{\text{eqn (16)}}\underbrace{\left(\frac{T_2}{T_1}\right)^{-1}}_{\text{eqn (13)}}} \quad (17)$$

It can be shown that (17) can be simplified to give:

$$\boxed{\frac{\rho_2}{\rho_1} = \frac{(\gamma+1)M_1^2}{2+(\gamma-1)M_1^2}} \quad (17A)$$

(no proof)

To get the change in stagnation pressure across the shock (i.e. $\frac{P_{02}}{P_{01}}$), recall from class lecture that

$$\boxed{\frac{P_0}{P} = \left(1+\left(\frac{\gamma-1}{2}\right)M^2\right)^{\gamma/(\gamma-1)}} \quad (18)$$

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THUS,

$$\frac{P_{02}}{P_{01}} = \left(\frac{P_{02}/P_2}{P_{01}/P_1}\right)\left(\frac{P_2}{P_1}\right)$$

FROM (18) AND (16), THIS BECOMES;

$$\boxed{\frac{P_{02}}{P_{01}} = \left[\frac{\left(1+\frac{\gamma-1}{2}\right)f(M_1^2)}{\left(1+\frac{\gamma-1}{2}\right)M_1^2}\right]^{\frac{\gamma}{\gamma-1}}\left[\frac{1+\gamma M_1^2}{1+\gamma f(M_1^2)}\right]} \quad (19)$$

* NOTE TABLE A.2 CAN BE USED ONLY FOR AIR (WHERE $\gamma = 1.4$) *

BEFORE LOOKING AT A FEW EXAMPLES

IT IS PHYSICALLY ENLIGHTENING

TO LOOK AT WHAT HAPPENS TO

THE RATIOS OF $\frac{P_2}{P_1}$, $\frac{T_2}{T_1}$, $\frac{\rho_2}{\rho_1}$ AND $\frac{P_{02}}{P_{01}}$,

AND TO M_2 AS M_1 BECOMES LARGE

(i.e. $M_1 \to \infty$). LET'S LIMIT ATTN. TO AIR (WHERE $\gamma = 1.4$);

FROM (12):

$$\boxed{M_2^2 (M_1^2 \to \infty) \to \frac{\gamma-1}{2\gamma} = \frac{.4}{2.8} = \frac{1}{7}}$$

FOR AIR

FROM (16) [w/ $f(M_1) \to 1/7$]

$$\boxed{\frac{P_2}{P_1} \to \frac{\gamma M_1^2}{1+\gamma\left(\frac{1}{7}\right)} \to \infty} \quad \text{AS } M_1 \to \infty$$

Similarly, from (13): 2/3/00 (12)

$$\boxed{\frac{T_2}{T_1} \rightarrow \frac{\left(\frac{\gamma-1}{2}\right) M_1^2}{1+\left(\frac{\gamma-1}{2}\right)\left(\frac{1}{\gamma}\right)} \rightarrow \infty} \text{ as } M_1 \rightarrow \infty$$

Finally, from (17A):

$$\boxed{\frac{\rho_2}{\rho_1} \rightarrow \frac{(\gamma+1)}{(\gamma-1)} = \frac{2.4}{.4} = 6} \text{ as } M_1 \rightarrow \infty$$

In order to show that the stagnation temp to remain constant across a normal shock, look at the energy eqn. across the shock;

By <u>defn.</u>, the <u>stagnation enthalpy</u>, h_0, is the enthalpy extant when a flow is accelerated or decelerated isentropically from or to a speed u and temp. T. Thus, for 1-D steady flow, an energy balance (in which frictional work and heat transfer are by assumption negligible) between a stagnation point and an arbitrary upstream (or downstream) location (where the

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LHS SPEED IS u AND TEMP. IS T)

YIELDS:

$$\boxed{\frac{1}{2}u^2 + C_pT = C_pT_0} \quad (20)$$

NOW, ACROSS A NORMAL SHOCK, THE ENERGY EQN READS

$$\frac{1}{2}u_1^2 + C_pT_1 = \frac{1}{2}u_2^2 + C_pT_2 \quad (21)$$

BUT, BY DEFINITION THE LHS OF (21) EQUALS THE STAGNATION ENTHALPY $h_{01} = C_pT_{01}$ WHILE THE RHS EQUALS $h_{02} = C_pT_{02}$.

(i.e., AT ANY POINT WHERE THE THE LOCAL FLOW SPEED IS u AND THE TEMPERATURE IS T, WE CAN ALWAYS ASSOCIATE (BY IMAGINING AN ISENTROPIC DECELERATION) A STAGNATION ENTHALPY.).

THUS (21) IMPLIES THAT

$$C_pT_{01} = C_pT_{02}$$

or

$$\boxed{T_{01} = T_{02}} \quad (22)$$

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EXAMPLE 1

A GAS w/ molar mass 28.9 (g/g-mol) and specific heat ratio of 1.4 discharges through a nozzle. A normal shock wave occurs @ a section where the Mach no. is 2.5, pressure is 40 kPa and temp is -20°C

Find downstream Mach no., pressure and temperature.

$M_1 = 2.5$	M_2 ?
$P_1 = 40\,kPa$	P_2 ?
$T_1 = 253\,K$	T_2 ?

$\gamma = 1.4$

$$R = 8.314\ J/gmol \cdot K \cdot \left(\frac{gmol}{28.9\,g}\right) \cdot \frac{10^3 g}{kg} = 287.7\ J/kg \cdot K$$

Let's use TABLE A.2 to solve:

AT $M_1 = 2.5$; TABLE A.2 gives

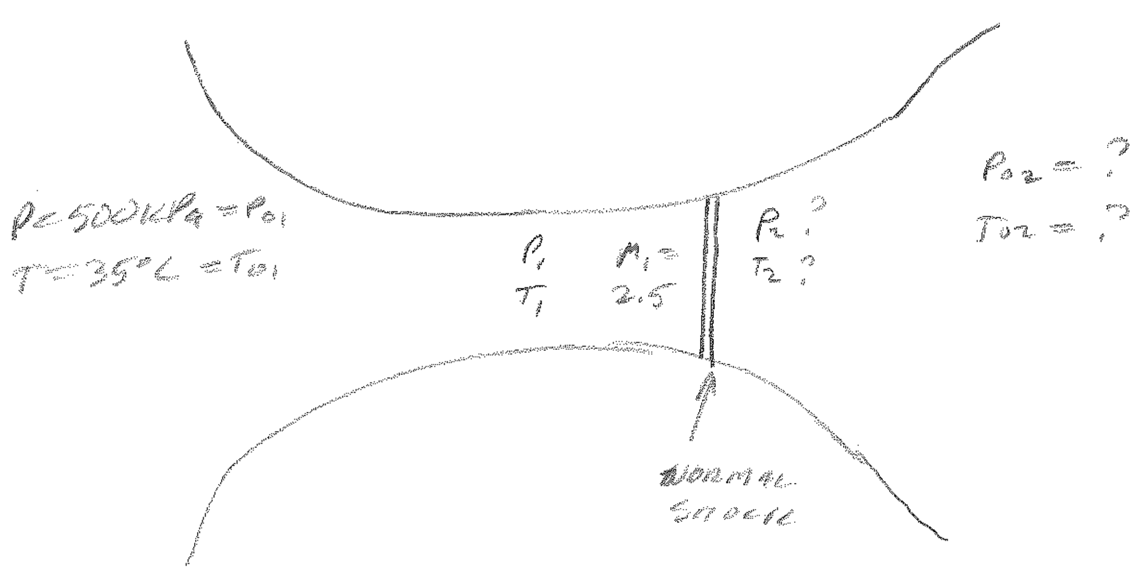
$$\frac{P_2}{P_1} = 7.125 \quad , \quad \frac{T_2}{T_1} = 2.137 \quad , \quad \boxed{M_2 = 0.513}$$

$$\Rightarrow \boxed{P_2 = (7.125)(40) = 285\,kPa}$$

$$\boxed{T_2 = (2.137)(253) = 540\,K}$$

EXAMPLE 2

AIR IS EXPANDED FROM A LARGE RESERVOIR IN WHICH PRESS. AND TEMP. ARE 500 KPa AND 35°C THROUGH A VARIABLE AREA DUCT. A NORMAL SHOCK OCCURS AT A POINT WHERE THE MACH NO. IS 2.5. WHAT'S THE PRESSURE AND TEMP. JUST DOWNSTREAM OF THE SHOCK? DOWNSTREAM OF THE SHOCK, THE FLOW IS BROUGHT TO REST IN ANOTHER RESERVOIR. WHAT'S THE PRESSURE AND TEMP. IN THIS DOWNSTREAM RESERVOIR? ASSUME ISENTROPIC 1-D FLOW.



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AGAIN, LET'S USE TABLE A.2:

ATTENTIVE STRATEGY:

1) USE ISENTROPIC RELATIONS TO GET P_1 & T_1 (GIVEN P_{01}, T_{01} & M_1).
2) USE NORMAL SHOCK RELNS TO GET P_2, T_2 (GIVEN P_1, T_1).
3) USE NORMAL SHOCK RELNS TO GET P_{02} (GIVEN P_{01}).

FIRST, WE KNOW THAT

$$\boxed{T_{02} = T_{01} = 308K}$$

FROM $\boxed{\text{TABLE A.1}}$ (WHICH AGAIN, CAN ONLY BE USED FOR AIR (SINCE $\gamma = 1.4$))

@ $M_1 = 2.5$:

$$\frac{P_{01}}{P_1} = 17.09$$

$$\frac{T_{01}}{T_1} = 2.25$$

$$\Rightarrow \boxed{P_1 = \frac{P_{01}}{17.09} = \frac{500\ kPa}{17.09} = 29.26\ kPa}$$

$$\boxed{T_1 = \frac{T_{01}}{2.25} = \frac{308K}{2.25} = 136.9\ K}$$

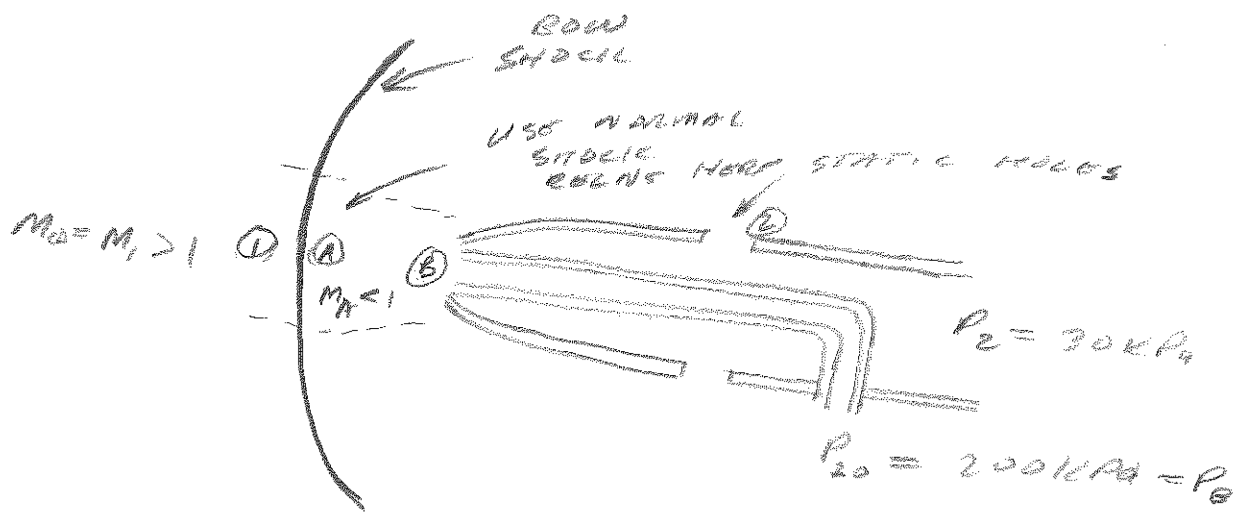
(SKIP TO TOP OF PAGE 17 FOR ANSWERS)

EXAMPLE 3 — PITOT TUBE IN SUPERSONIC FLOW

A PITOT TUBE (SHOWN) IS USED TO MEASURE THE FREE-STREAM MACH NO., M_∞. THE PRESSURES MEASURED ARE SHOWN. IT IS KNOWN EXPERIMENTALLY FOR THIS PITOT TUBE THAT THE PRESSURE COEFFICIENT, C_p, DEFINED AS

$$C_p = \frac{P_2 - P_\infty}{\frac{1}{2}\rho_\infty U_\infty^2}$$

IS EQUAL TO 0.1. WHAT IS THE FREE STREAM MACH NO.?



ANSWER

FOR PITOT TUBE PROBLEMS, WE USE THE NORMAL SHOCK RELATIONS TO ESTIMATE PROPERTY CHANGES ACROSS THE LEADING PORTION OF THE BOW SHOCK. WE ALSO ASSUME THAT SINCE THE FLOW BEHIND THIS PART OF THE SHOCK IS

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IS SUBSONIC THAT THE DECELERATION FROM PT. (A) TO PT. (B) IS ISENTROPIC.

LET'S REWRITE C_p AS

$$C_p = \frac{P_2 - P_\infty}{\frac{\gamma}{2} P_\infty M_\infty^2} = \left(\frac{2}{\gamma}\right)\left(\frac{1}{M_\infty^2}\right)\left(\frac{P_2}{P_\infty} - 1\right)$$

* NOTE P_2 IS THE STATIC PRESSURE AT THE STATIC PRESSURE PORTS.

TRY TO COME UP W/ A SOLN APPROACH!

1) $\frac{P_{20}}{P_2} = f(M_2)$ = isentropic relation where M_2 is Mach No. @ PT. (2)

⇒ CAN BACK OUT M_2 (ASSUMES ISENTROPIC FLOW FROM PT (B) TO (2))

2) $\frac{P_A}{P_{20}} = f^{-1}(M_A)$ = isentropic relation

3) $\frac{P_A}{P_2} = \left(\frac{P_A}{P_{20}}\right)\left(\frac{P_{20}}{P_2}\right) = f^{-1}(M_A)\, f(M_2)$ (REDUNDANT W/ 2))

4) $M_A = g(M_\infty)$ = eqn (2) = normal shock reln. between M_A AND M_∞

5) 3) + 4) ⇒ $\frac{P_A}{P_2} = f^{-1}(g(M_\infty))\, f(M_2)$

⇒

$$P_A = P_2\, f^{-1}(g(M_\infty))\, f(M_2)$$

6) $\frac{P_A}{P_\infty} = g_1(M_\infty)$ = normal shock reln. between P_A & P_∞

7) $5) + 6) \Rightarrow \quad P_\infty = \dfrac{P_A}{g_P(M_\infty)}$

$$P_\infty = \frac{P_2 f^{-1}\left(g(M_\infty)\right) f(M_2)}{g_P(M_\infty)}$$

8)

$$\boxed{C_P = \frac{2}{\gamma}\left(\frac{1}{M_\infty^2}\right)\left[\frac{P_2}{P_2 \left(\dfrac{f^{-1}(g(M_\infty))\, f(M_2)}{g_P(M_\infty)}\right)} - 1\right]} \quad (*)$$

$\Rightarrow$ solve 8) for $M_\infty \Rightarrow$ soln

now list the functions f, g, & g_P !

$$\frac{P_0}{P} = f(M) = \left(\frac{\frac{1}{\gamma-1} + \frac{1}{2}M^2}{\frac{1}{\gamma-1}}\right)^{\frac{\gamma}{\gamma-1}} \quad = \text{ISENTROPIC RELATION w/ } M_0 = 0$$

$$M_A = g(M_\infty) = \frac{\left(M_\infty^2 + \frac{2}{\gamma-1}\right)}{\left(\frac{2\gamma}{\gamma-1}M_\infty^2 - 1\right)} \quad = \text{NORMAL SHOCK RELATION (eqn (12))}$$

$$\frac{P_A}{P_\infty} = g_P(M_\infty) = \frac{2\gamma M_\infty^2 - (\gamma-1)}{\gamma+1} \quad = \text{NORMAL SHOCK RELN. (eqn. 16A)}$$

SOLVE $(*)$ BY ITERATION — YOU SHOULD GET FOLLOWING :

$$\boxed{M_\infty = 1.93}$$

NOW FROM TABLE A.2:

@ $M_1 = 2.5$: $\frac{P_2}{P_1} = 7.125$ $\frac{T_2}{T_1} = 2.137$ $\frac{P_{02}}{P_{01}} = 0.499$

$\Rightarrow$ $\boxed{P_2 = (7.125)(29.26) = 208.5\ kPa}$

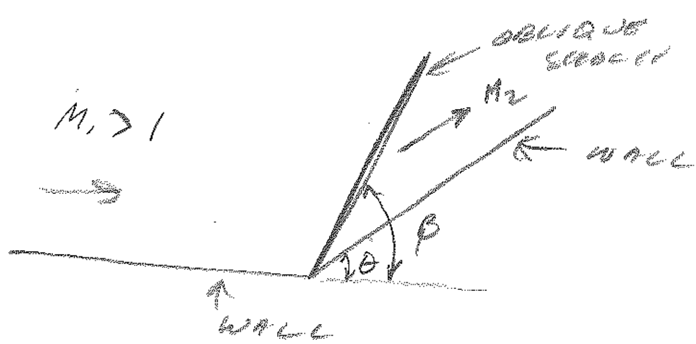
$T_2 = (2.137)(136.9) = 292.6K$

$P_{02} = (0.499)(500) = 250kPa$

II) OBLIQUE SHOCKS

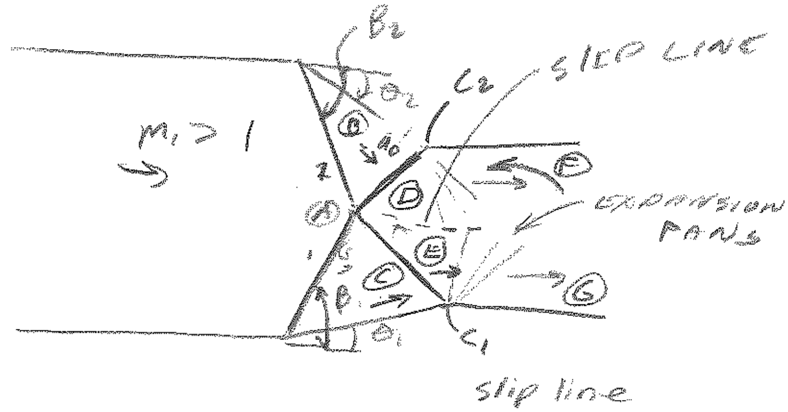
EXAMPLE

SUPERSONIC FLOW INTO A COMPRESSIVE CORNER



ONCE FLOW PASSES THROUGH THE OBLIQUE SHOCK, IT HAS TURNED UPWARD BY AN ANGLE θ.

EXAMPLE 2



$\theta_1 \neq \theta_2$

IN THIS FLOW, THE SUPERSONIC FLOW PASSES INTO TWO OPPOSING COMPRESSING CORNERS (= CORNERS WHICH TURN THE FLOW INTO ITSELF — i.e., CORNERS HAVING AN INCLUDED ANGLES (θ_1 & θ_2 IN THIS CASE) LESS THAN 180°). TWO OBLIQUE SHOCKS (1 & 2) FORM AT ANGLES β_1 & β_2 RELATIVE TO HORIZONTAL. THE SHOCKS MEET AT PT. A WHERE, DUE TO THE FACT THAT THE FLOWS IN REGIONS B & C ARE AGAIN TURNED COMPRESSIVELY (BY A SLIP-LINE EMANATING FROM PT A), TWO MORE OBLIQUE SHOCKS FORM. THE FOUR SHOCKS FORM AN 'X' PATTERN W/ CENTER AT PT. A.

THE FLOW IN REGION B IS PARALLEL TO THE UPPER INCLINED WALL; SIMILARLY, FLOW IN REGION C IS PARALLEL TO THE

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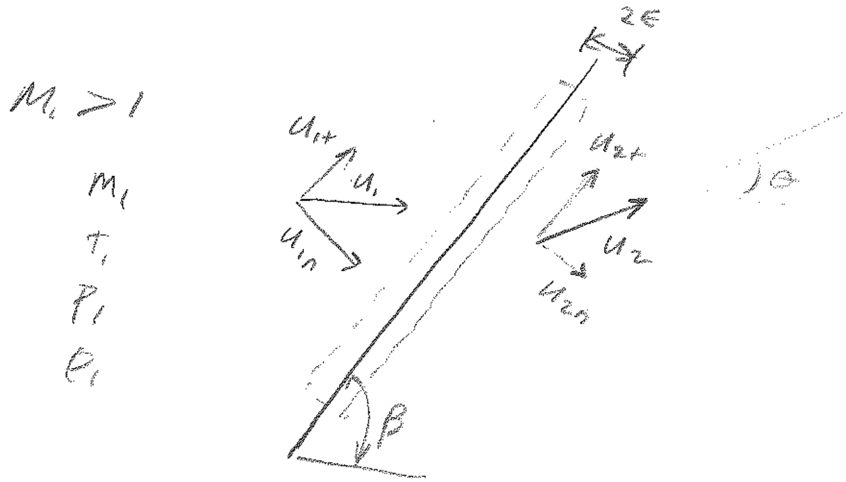
LOWER INCLINED WALL. THE SECOND SET OF OBLIQUE SHOCKS (WHICH NOMINALLY END AT THE CORNERS C_1 AND C_2) FORM BECAUSE THE SLIP-LINE BEHAVES ESSENTIALLY LIKE A VIRTUAL WALL W/IN THE FLOW, FORCING THE FLOW LEAVING THE SECOND SET OF SHOCKS TO FLOW PARALLEL TO THE SLIP-LINE (IN REGIONS D & E).

FINALLY, THE FLOWS IN D & E ARE TURNED THROUGH THE EXPANSION FANS EMANATING FROM C_1 & C_2 -- ONCE THE FLOWS PASS THROUGH THE FANS, THEIR DIRECTIONS ARE PARALLEL TO THE DOWNSTREAM (PARALLEL) DUCT WALLS. WE WILL SOON BE ABLE TO ANALYZE THIS FLOW COMPLETELY

2/03/00 (20)

B) DERIVATION OF WORKING EQUATIONS



APPROACH: SHOW THAT TANGENTIAL VELOCITY COMPONENTS (u_{1t} & u_{2t}) ARE EQUAL (USING CONS. OF TANGENTIAL MOMENTUM), THEN SHOW THAT CONSERV. OF MASS, MOMENTUM (NORMAL COMPONENT), AND ENERGY HAVE THE SAME FORM AS THE EQUATIONS DERIVED FOR NORMAL SHOCKS — HERE, HOWEVER, THE NORMAL VELOCITY COMPONENTS (u_{n1} & u_{n2}) REPLACE THE TOTAL VELOCITIES (u_1 & u_2) USED IN DERIVING THE NORMAL SHOCK RELATIONS. ONCE THIS IS SHOWN, THEN ALL OF THE PREVIOUSLY DERIVED NORMAL SHOCK EQNS. CAN BE WRITTEN IN TERMS OF u_{n1} & u_{n2} — THAT COMPLETES THE DERIVATION.

2/03/00 (21)

TO PROVE THAT $u_{1t} = u_{2t}$, APPLY CONS. OF LINEAR MOM. IN TANGENTIAL DIRECTION:

$$\oint_S \rho u_i (\underline{u}\cdot\hat{n})\, dS = \oint_S \sigma_{ij} n_j\, dS$$

(S = surface surrounding CV)

$$= \oint -P n_i\, dS$$

(Choose $i=1$ direction to be the tangent direction)

$$= -P_{TOP}(2\epsilon)b + P_{BOTTOM}(2\epsilon)b$$

b = width of control volume (into page)

$$\oint_S \rho u_t (\underline{u}\cdot\underline{n})\, dS = \int_{S_1} \rho \overset{u_{1t}}{\cancel{u_t}} \overset{(-u_{1n})}{\cancel{(\underline{u}\cdot\underline{n})}}\, dS + \int_{S_2} \rho \overset{u_{2t}}{\cancel{u_t}} \overset{(u_{2n})}{\cancel{(\underline{u}\cdot\underline{n})}}\, dS$$

↓ use 1-D flow assumption

$$= -\rho_1 u_{1t} u_{1n} S_1 + \rho_2 u_{2t} u_{2n} S_2$$

NOW SINCE $S_1 (= bw) \gg 2\epsilon b$ (i.e., THE TANGENTIAL LENGTH OF THE CONTROL VOLUME w IS MUCH GREATER THAN THE THICKNESS ϵ), THEN PRESSURE TERMS ARE NEGLIGIBLE AND MOMENTUM EQN SIMPLIFIES TO

$$\boxed{\rho_1 u_{1t} u_{1n} \cancel{S_1} = \rho_2 u_{2t} u_{2n} \overset{S_1}{\cancel{S_2}}} \quad (23)$$

2/03/00 (22)

NOW, WE CAN APPLY CONS. OF MASS TO SHOW THAT $\rho_1 u_{1n} = \rho_2 u_{2n}$;

$$\frac{dM_{sys}}{dt} = \oint \rho (u \cdot n)\, ds = 0$$

$$\Rightarrow \int_{S_1} \rho (\overset{-u_{1n}}{u \cdot n})\, ds + \int_{S_2} \rho (\overset{u_{2n}}{u \cdot n})\, ds = 0$$

USE UNIFORM FLOW ASSUMPTION $\rightarrow$

$$-\rho_1 u_{1n} S_1 + \rho_2 u_{2n} \overset{S_1}{S_2} = 0$$

$$\Rightarrow \boxed{\rho_1 u_{1n} = \rho_2 u_{2n}} \quad (24)$$

THUS, USING (24) IN (23), WE FINALLY SHOW THAT

$$\boxed{u_{1t} = u_{2t}} \quad (25)$$

TO GET THE THIRD NEEDED EQN., APPLY CONS. OF LIN. MOMENTUM IN THE NORMAL DIRECTION;

$$\frac{d}{dt}\int_{MC} \rho u_n \, dV = \overset{0 \text{ steady}}{\cancel{\frac{d}{dt}}\int_{CV} \rho u_n \, dV} + \oint_S \rho u_n (u \cdot n)\, ds$$

$$= \Sigma F_n$$

$$= \oint_S \sigma_{ij} n_j \, ds$$

let index i correspond to normal direction

CONSIDER EACH TERM INDIVIDUALLY IN LAST EQN.;

$$\oint_S \rho u_n (\mathbf{u}\cdot\mathbf{n})\, dS = \int_{S_1} \rho \overset{u_{1n}}{u_n} (\overset{(-u_{1n})}{\mathbf{u}\cdot\mathbf{n}})\, dS + \int_{S_2} \rho \overset{u_{2n}}{u_n} (\overset{(u_{2n})}{\mathbf{u}\cdot\mathbf{n}})\, dS$$

(UNIF. FLOW ASSUMPTION) $\Rightarrow$

$$= -\rho_1 u_{1n}^2 S_1 + \rho_2 u_{2n}^2 \overset{S_1}{\cancel{S_2}}$$

$$\oint_S \sigma_{ij} n_j\, dS = \oint_S -P n_1\, dS = -\int_{S_1} P \overset{(-1)}{n_1}\, dS - \int_{S_2} P \overset{(1)}{n_1}\, dS$$

(neglect viscous terms in σ_{ij})

$$= P_1 S_1 - P_2 \overset{S_1}{\cancel{S_2}}$$

THUS,

$$\boxed{\rho_2 u_{2n}^2 + P_2 = \rho_1 u_{1n}^2 + P_1} \quad (26)$$

THE FINAL EQN. WE NEED IS THE ENERGY BALANCE. USING THE SAME APPROACH AS ABOVE, WE FIND THAT

$$\boxed{c_p T_1 + \frac{u_{1n}^2}{2} = c_p T_2 + \frac{u_{2n}^2}{2}} \quad (27)$$

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NOW COMPARING EQNS. (24), (26) & (27) WITH THE BALANCE EQNS. OBTAINED ACROSS A NORMAL SHOCK, WE SEE THAT THEY ARE OF IDENTICAL FORM W/ u_{1n} REPLACING u_1 & u_{2n} REPLACING u_2. THUS, IF WE DEFINE M_{n_1} AND M_{n_2} AS

$$\boxed{M_{n_1} = \frac{u_{n_1}}{a_1}} \qquad (28)$$

$$\boxed{M_{n_2} = \frac{u_{n_2}}{a_2}} \qquad (29)$$

WE OBTAIN

FROM (12)

$$\boxed{M_{n_2}^2 = \frac{\left[M_{n_1}^2 + \left(\frac{2}{\gamma-1}\right)\right]}{\left[\left(\frac{2\gamma}{\gamma-1}\right)M_{n_1}^2 - 1\right]}} \qquad (30)$$

(OBLIQUE SHOCK)

FROM (16a)

$$\boxed{\frac{p_2}{p_1} = \frac{2\gamma M_{n_1}^2 - (\gamma-1)}{(\gamma+1)}} \qquad (31)$$

OBLIQUE SHOCK

(B) ⇒

$$\boxed{\frac{T_2}{T_1} = \frac{1 + \left(\frac{\gamma - 1}{2}\right) M_{in}^2}{1 + \left(\frac{\gamma - 1}{2}\right) f(M_{in}^2)} \qquad (32)}$$

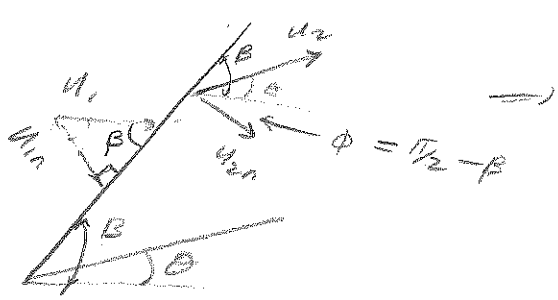
$$\boxed{f(M_{in}^2) = \frac{(\gamma - 1) M_{in}^2 + 2}{2\gamma M_{in}^2 - (\gamma - 1)} \qquad (33)}$$

OBLIQUE SHOCK

(17A) ⇒

$$\boxed{\frac{\rho_2}{\rho_1} = \frac{(\gamma + 1) M_{in}^2}{2 + (\gamma - 1) M_{in}^2} \qquad (34)}$$

IN ORDER TO COMPLETE THE DERIVATION WE NEED TO RELATE u_{n1} & u_{n2} to THE ANGLES β (shock angle) AND θ (turning angle).



FROM FIGURE

$$\boxed{u_{1n} = u_1 \sin\beta} \qquad (35)$$

$$u_{2n} = u_2 \cos(\phi + \theta) = u_2 \cos\left(\tfrac{\pi}{2} - \beta + \theta\right)$$

$$\boxed{u_{2n} = u_2 \sin(\beta - \theta)} \qquad (36)$$

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THUS

$$M_{1n} = \frac{u_{1n}}{a_1} = \frac{u_1}{a_1}\sin\beta = M_1 \sin\beta \quad (37)$$

$$M_{2n} = \frac{u_{2n}}{a_2} = \frac{u_2}{a_2}\sin(\beta-\theta) = M_2 \sin(\beta-\theta) \quad (38)$$

SO FINAL WORKING EQNS ARE OBTAINED BY SUBSTITUTING (37) & (38) INTO (30), (31), (32), (33) & (34):

(30) ⇒
$$\boxed{M_2^2 \sin^2(\beta-\theta) = \frac{\left(M_1^2 \sin^2\beta + \frac{2}{\gamma-1}\right)}{\left[\left(\frac{2\gamma}{\gamma-1}\right)M_1^2 \sin^2\beta - 1\right]}} \quad (39)$$

(31) ⇒
$$\boxed{\frac{P_2}{P_1} = \frac{2\gamma M_1^2 \sin^2\beta - (\gamma-1)}{\gamma+1}} \quad (40)$$

(32) ⇒
$$\boxed{\frac{T_2}{T_1} = \frac{1 + \frac{(\gamma-1)}{2} M_1^2 \sin^2\beta}{1 + \left(\frac{\gamma-1}{2}\right) f\left(M_1 \sin^2\beta\right)}} \quad (41)$$

(34) ⇒
$$\boxed{\frac{\rho_2}{\rho_1} = \frac{(\gamma+1)\left(M_1^2\right)\sin^2\beta}{2 + (\gamma-1)M_1^2 \sin^2\beta}} \quad (42)$$

IN ORDER TO SOLVE OBLIQUE SHOCK PROBLEMS, WE NEED A RELATIONSHIP BETWEEN θ, β and M_1 -- M_1 & θ are generally KNOWN and, AS WE'LL SHOW, CAN THUS BE USED TO DETERMINE THE SHOCK ANGLE β.

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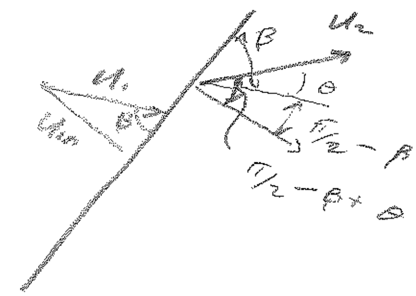
DERIVATION OF RELATIONSHIP BETWEEN

θ, M_1 & β:



FROM FIG.

$$\frac{u_{1n}}{u_{1t}} = \tan\beta$$

$$\frac{u_{2n}}{u_{2t}} = \tan\left(\frac{\pi}{2} - \left(\frac{\pi}{2} - \beta + \theta\right)\right) = \tan(\beta - \theta)$$

BUT $u_{1t} = u_{2t}$ (AS SHOWN ABOVE)

ALSO, FROM CONT'Y

$$\frac{u_{2n}}{u_{1n}} = \frac{\rho_1}{\rho_2} = \frac{u_{2n}/u_{2t}}{u_{1n}/u_{1t}} = \frac{\tan(\beta-\theta)}{\tan\beta}$$

OR USING (42) $\Rightarrow$

$$\boxed{\frac{\tan(\beta-\theta)}{\tan(\beta)} = \frac{2 + (\gamma-1) M_1^2 \sin^2\beta}{(\gamma+1) M_1^2 \sin^2\beta}} \qquad (43)$$

NOW NOTE THAT

$$\tan(\beta-\theta) = \frac{\tan\beta - \tan\theta}{1 + \tan\beta\tan\theta}$$

$\Rightarrow$ (43) $\Rightarrow$ $1 - \tan\theta/\tan(\beta) = X(1 + \tan\beta\tan\theta)$

$= X + X\tan\beta\tan\theta$

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WHERE $\boxed{X = \frac{2+(\gamma-1)M_1^2\sin^2\beta}{(\gamma+1)M_1^2\sin^2\beta}}$ (44A)

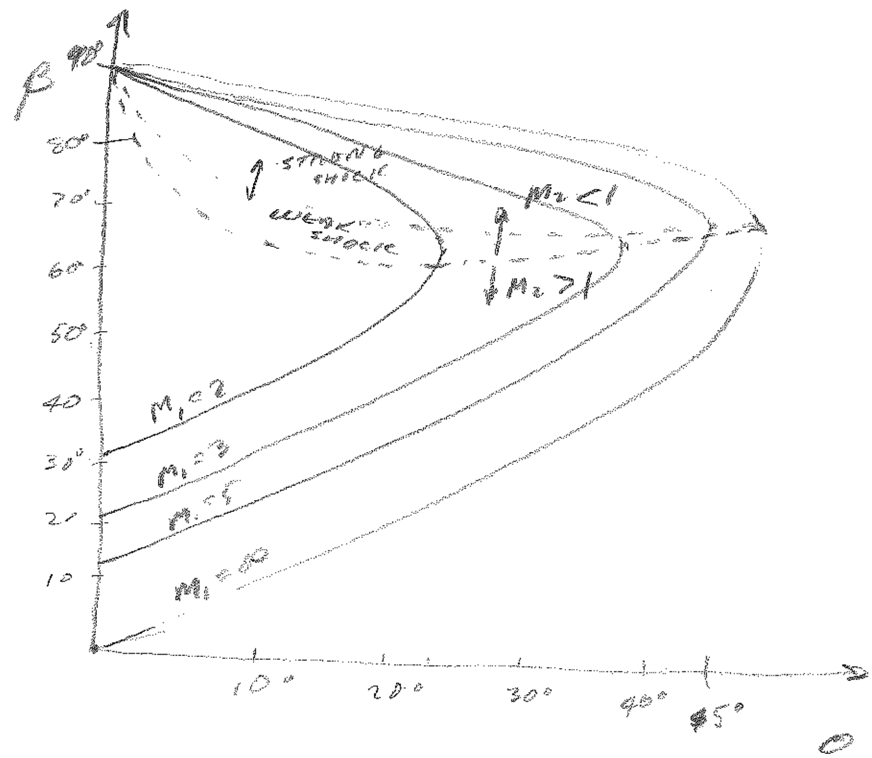
SOLVING LAST EQN ON PREVIOUS PAGE FOR $\tan\theta$ WE GET:

$$\tan\theta(-1 - X\tan^2\beta) = X\tan\beta - \tan(\beta)$$

$$\boxed{\tan\theta = \frac{\tan\beta(1-X)}{(1+X\tan^2\beta)}} \qquad (44)$$

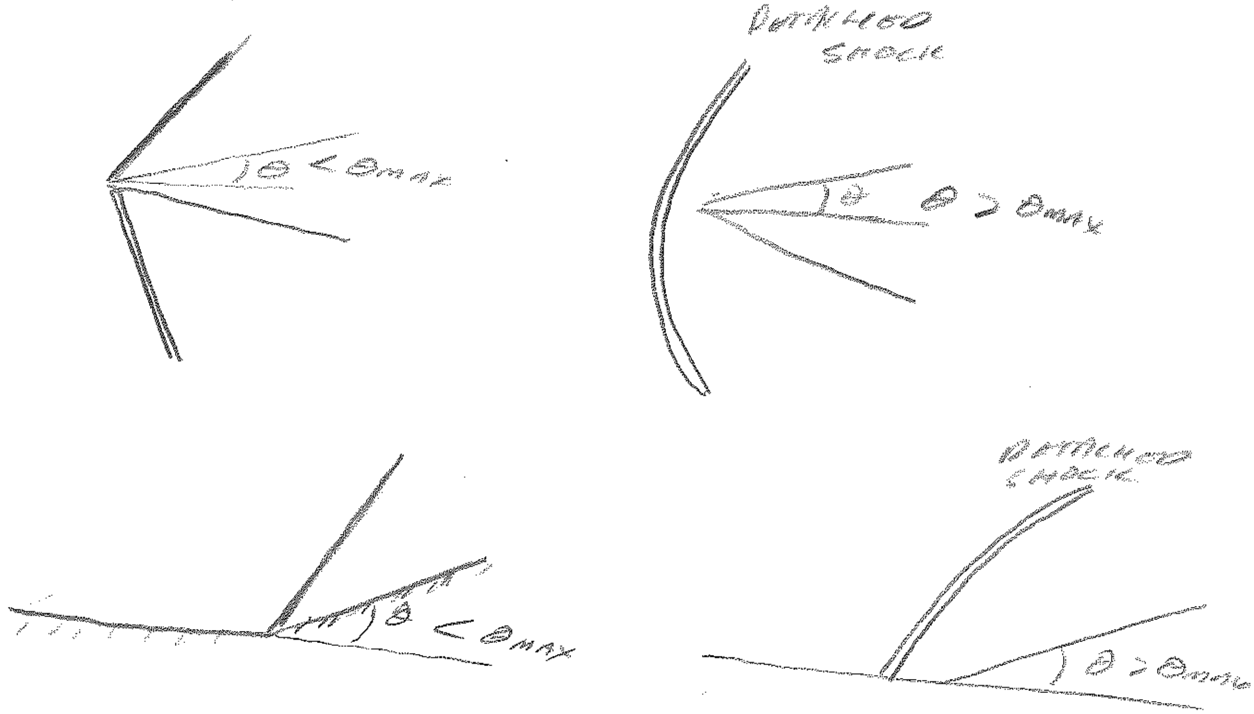
EQN (44) (& 44A) IS THE DESIRED RELATIONSHIP. IT GIVES θ AS A FN. OF M_1 AND β

AS DISCUSSED IN YOUR TEXT ON PGS 107–109, THERE ARE SEVERAL PHYSICAL IMPLICATIONS ASSOCIATED W/ EQN (44).



IMPLICATIONS OF EQN. (44)

1) FOR ANY GIVEN M_1 THERE'S A MAXIMUM VALUE OF $\theta = \theta_{max}$ ALLOWING SOLN. OF (44). FOR DEFLECTION ANGLES $> \theta_{max}$, A STRAIGHT OBLIQUE SHOCK CANNOT EXIST (CORRESPONDING TO NO SOLN. OF (44)). FOR $\theta > \theta_{max}$, THE SHOCK MOVES AWAY FROM THE TURNING SURFACE. EXAMPLES (FROM TEXT):



2) FOR ANY $\theta < \theta_{max}$ AND ANY GIVEN M_1, THERE ARE TWO VALUES OF β THAT SATISFY (44). THE <u>LARGER</u> β CORRESPONDS TO A <u>STRONG</u> <u>SHOCK</u> WHILE THE <u>SMALLER</u> β GIVES THE <u>WEAK</u> <u>SHOCK</u> ANGLE. THE WEAK SHOCK SOLN. TENDS TO OCCUR IN MOST CASES. INDEED THE WEAK

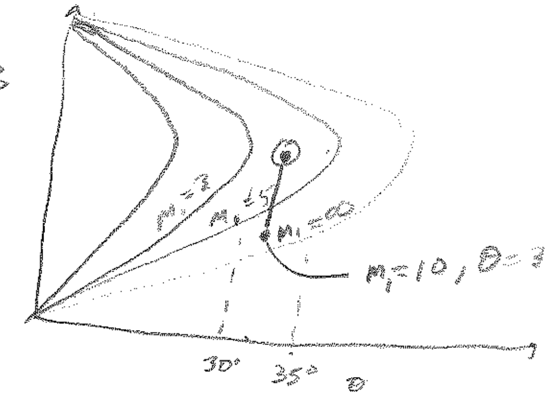
SHOCK SOLN IS FAVORED FROM AN ENTROPY STANDPOINT (THE ENTROPY CHANGE ACROSS A WEAK OBLIQUE SHOCK IS SMALLER THAN OVER A STRONG SHOCK). THE UPSTREAM MACH NO., M_2, IS SUPERSONIC BEHIND A WEAK SHOCK (EXCEPT NEAR θ_{max} AS SHOWN IN FIG. 4.5 IN TEXT.), AND ALWAYS SUBSONIC BEHIND A STRONG SHOCK.

3) IF $\theta = 0°$, (i.e., NO DEFLECTION) (44) HAS SOLNS $\beta = \frac{\pi}{2}$ (CORRESPONDING TO A STRONG SHOCK) AND $\beta = \sin^{-1}\frac{1}{M_1} = \mu$. (WE WILL DISCUSS THE LATTER SOLN. NEXT LECTURE.)

4) FOR FIXED θ (DEFLECTION ANGLE), THE VALUE OF β INCREASES AS M_1 DECREASES. ONCE M_1 DROPS BELOW SOME CRITICAL VALUE, (44) HAS NO SOLN, AND THE ATTACHED OBLIQUE SHOCK DETACHES.

EXAMPLE

LET $\theta = 32°$
SHOW GRAPHICALLY THAT AS M_1 DECREASES FROM SAY 10, A CRITICAL M_1 IS EVENTUALLY REACHED WHERE NO SOLN. EXISTS

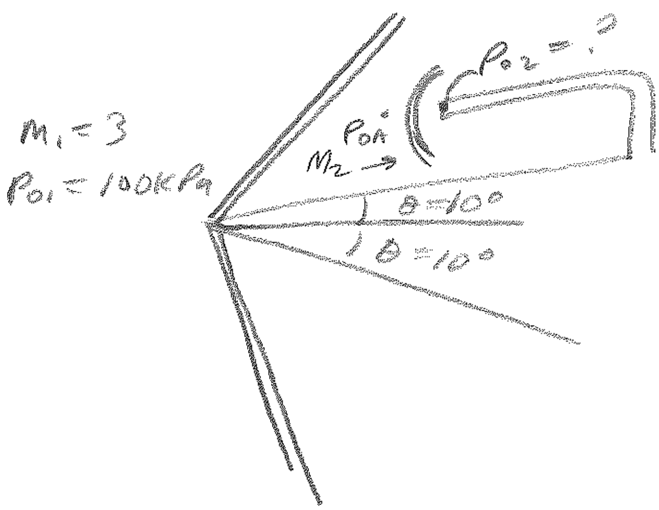


FROM GRAPH, SOMEWHERE BETWEEN $M_1=5$ AND $M_1=3$, A SOLN IS NO LONGER POSSIBLE

(NOTE $M_1=3$ CURVE NEVER INTERSECTS $\theta=32°$)

EXAMPLE

A WEDGE IS PLACED IN A FLOW WHERE $M_1 = 3$ AND THE FREE STREAM STAGNATION PRESSURE IS $P_{01} = 100\,kPa$. IF A PITOT TUBE IS LOCATED BEHIND THE OBLIQUE SHOCK AS SHOWN, WHAT STAGNATION PRESSURE WILL IT READ?



SOLN. STRATEGY

1) GIVEN M_1 & θ, get β (ASSUME WEAK SHOCK) FROM FIGURE.
2) GIVEN β and M_1, get $M_{2n} = fn(M_{1n})$
   $(M_{1n} = M_1 \sin\beta)$
3) GIVEN M_{2n}, get $M_2 = M_{2n}/\sin(\beta - \theta)$
4) GIVEN M_2, get $\dfrac{P_{02}}{P_{0A}} = fn(M_2) =$ normal shock eqn. for $\dfrac{P_{02}}{P_{01}}$.

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5) $\frac{P_{0A}}{P_{01}} = fn(M_{1n})$ = normal shock eqn for $\frac{P_{02}}{P_{01}}$ in terms of $\underline{M_{1n}}$.

$\Rightarrow P_{0A} = P_{01}\, fn(M_{1n})$

6) From 4) and 5) solve for P_{02}:

$$P_{02} = P_{0A}\, fn(M_2)$$
$$= P_{01}\, fn(M_{1n})\, fn(M_2)$$

<u>Soln</u>:

1) For $M_1 = 3$ $\theta = 10°$ $\Rightarrow \beta \approx 27.5°$ (From chart on inside cover of text)

2) $M_{2n} = f(M_{1n})$ $\quad M_{1n} = M_1 \sin\beta$
$= 3 \sin(27.5°) = \underline{1.385}$

$\Rightarrow M_{2n} = 0.7483$ (From Table A.2 using $M = 1.385$)

3) $\Rightarrow M_2 = \frac{M_{2n}}{\sin(\beta-\theta)} = \frac{.7483}{\sin(17.5°)} = \underline{2.49}$

4) $\frac{P_{02}}{P_{0A}} = fn(M_2) = \underline{8.52}$ (From Tbl. A.2 using $M_1 = 2.49$)

5) $\frac{P_{0A}}{P_{01}} = fn(M_{1n}) \approx \underline{0.963}$ (" " " " " = 1.385)

$\Rightarrow P_{0A} = P_{01}(.963)$
$= 96.3$ kPa

6) $\boxed{P_{02} = (8.52)(P_{0A}) = 820.5 \text{ kPa}}$