①

CONSERVATION OF LINEAR MOMENTUM

(DIFFERENTIAL FORM).

PROCEDURE

1) APPLY NEWTON'S 2nd LAW TO CUBICAL, INFINITESIMAL FLUID PTCL.

(A) $\delta m \vec{a} = \Sigma \vec{F}$ $\qquad \delta m = \rho \delta x \delta y \delta z$ (B)

2) USE TAYLOR EXPANSION ABOUT PTCL'S INITIAL POSITION (x, y, z) TO DETERMINE PTCL. ACCELERATION:

$$\vec{a} = \frac{\vec{u}(x+\Delta x, y+\Delta y, z+\Delta z, t+\Delta t) - \vec{u}(x,y,z,t)}{\Delta t}$$

$$\Rightarrow \vec{a} = \left[\frac{\vec{u}(x,y,z,t)}{\Delta t} + \frac{\partial \vec{u}}{\partial x}\overbrace{\frac{\Delta x}{\Delta t}}^{u} + \frac{\partial \vec{u}}{\partial y}\overbrace{\frac{\Delta y}{\Delta t}}^{v} + \frac{\partial \vec{u}}{\partial z}\overbrace{\frac{\Delta z}{\Delta t}}^{w} + \frac{\partial \vec{u}}{\partial t}\overbrace{\frac{\Delta t}{\Delta t}}^{1} - \frac{\vec{u}(x,y,z,t)}{\Delta t}\right]$$

$$\boxed{\vec{a} = u\frac{\partial \vec{u}}{\partial x} + v\frac{\partial \vec{u}}{\partial y} + w\frac{\partial \vec{u}}{\partial z} + \frac{\partial \vec{u}}{\partial t} \equiv \frac{D\vec{u}}{Dt} = \text{MAT'L DERIV. OF } \vec{u}}$$ (1)

3) WRITE DOWN BODY FORCE ACTING ON PTCL:

$$\boxed{\delta m \vec{g} = \rho \delta x \delta y \delta z \vec{g}}$$ (2)

4) EXPRESS SURFACE FORCES IN TERMS OF NORMAL STRESSES AND TANGENTIAL STRESSES.

A) SIGN CONVENTIONS / NOMENCLATURE

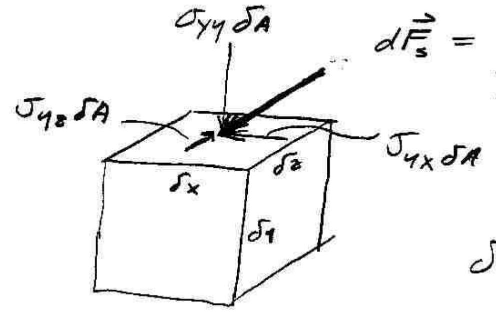
- σ_{xx} = normal stress in x-directio
- τ_{yx} = shear stress acting in X-DIR. ON FACE W/ NORMAL IN y-dir.

  (first subscript refers to normal direction, 2nd refers to direction stress acts in)

etc.

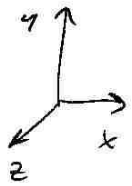
- A STRESS ACTING IN A POSITIVE COORDINATE DIRECTION ON A FACE W/ A NORMAL IN A POSITIVE COORDINATE DIR. IS POSITIVE
- A STRESS ACTING IN A NEGATIVE COORD. DIRECTION ON A FACE W/ A NORMAL DIRECTION IN A NEGATIVE COORD. DIR. IS POSITIVE.

EXAMPLE



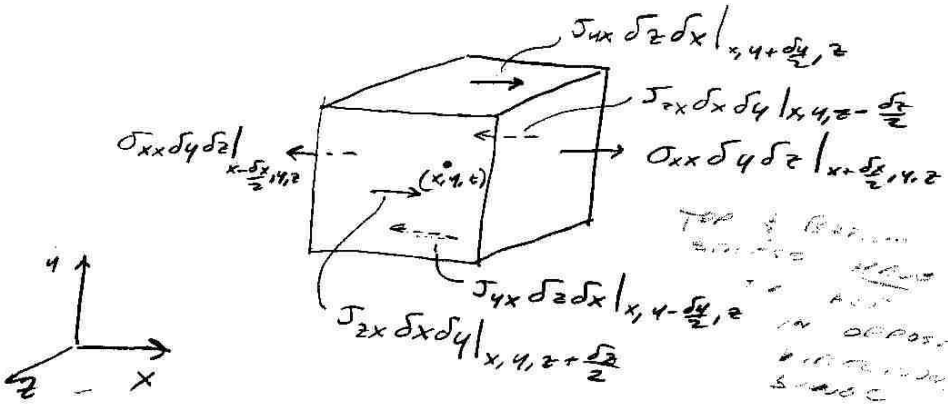
$d\vec{F}_s$ = SURFACE FORCE ACTING ON TOP SURFACE

$\delta A = \delta z\,\delta x$



HERE, σ_{yy}, τ_{yx} AND τ_{yz} ARE ALL NEGATIVE

LET'S NOW DETERMINE NET SURFACE FORCE IN X-DIRECTION. THE NET FORCES IN Y- AND Z-DIR WILL FOLLOW BY INSPECTION



TOP & BOTTOM [illegible] ACT IN OPPOSITE DIRECTIONS SINCE AS $\delta y \to 0$ $\Longrightarrow$ CAN'T HAVE A NON-ZERO [illegible]

NOTE, ALL SURFACE STRESSES SHOWN ARE POSITIVE. (WE DEFINE COORDINATES SO THAT THIS IS TRUE.) NOTE TOO THAT STRESSES ON OPPOSING FACES MUST ACT IN OPPOSITE DIRECTIONS. THIS IS SHOWN BY SHRINKING THE VOLUME OF THE CUBE TO NEAR-ZERO AND DOING A FORCE BALANCE, SAY IN THE X-DIRECTION:

$$\rho\,\delta x\,\delta y\,\delta z\,a_x = \left(\sigma_{xx}\big|_{x+\frac{\delta x}{2}} - \sigma_{xx}\big|_{x-\frac{\delta x}{2}}\right)\delta y\,\delta z + \left(\tau_{yx}\big|_{y+\frac{\delta y}{2}} - \tau_{yx}\big|_{y-\frac{\delta y}{2}}\right)\delta x\,\delta z + \left(\tau_{zx}\big|_{z+\frac{\delta z}{2}} - \tau_{zx}\big|_{z-\frac{\delta z}{2}}\right)\delta x\,\delta y + \rho g_x\,\delta x\,\delta y\,\delta z \tag{2}$$

I, II, III

As $\delta V = \delta x\,\delta y\,\delta z \to 0$, the acceleration and body force terms in (2) BECOME MUCH SMALLER THAN THE SURFACE FORCE TERMS. THUS, IN THE SEPARATE CASES WHERE: 1) $\tau_{yx} = \tau_{zx} = 0$, $\boxed{\sigma_{xx}\big|_{x+\frac{\delta x}{2}} = \sigma_{xx}\big|_{x-\frac{\delta x}{2}}}$; 2) $\sigma_{xx} = \tau_{yx} = 0$, $\boxed{\tau_{zx}\big|_{z+\frac{\delta z}{2}} = \tau_{zx}\big|_{z-\frac{\delta z}{2}}}$; 3) $\sigma_{xx} = \tau_{zx} = 0$, $\boxed{\tau_{yx}\big|_{y+\frac{\delta y}{2}} = \tau_{yx}\big|_{y-\frac{\delta y}{2}}}$

WE CAN USE (2) TO DETERMINE THE NET SURFACE FORCE IN X-DIRECTION ACTING ON THE CUBE. FROM (II) IN EQN. (2) WE HAVE

$$\left(\sigma_{xx}\big|_{x+\frac{\delta x}{2}} - \sigma_{xx}\big|_{x-\frac{\delta x}{2}}\right)\delta y\,\delta z = \left[\sigma_{xx}\big|_x + \frac{\partial}{\partial x}\sigma_{xx}\big|_x \frac{\delta x}{2} - \left(\sigma_{xx}\big|_x - \frac{\partial}{\partial x}\sigma_{xx}\right.\right.$$

$$\cdot\,\delta y\,\delta z$$

$$= \frac{\partial}{\partial x}\sigma_{xx}\,\delta x\,\delta y\,\delta z\Big|_{x,y,z}$$

$$\left(\tau_{yx}\big|_{y+\frac{\delta y}{2}} - \tau_{yx}\big|_{y-\frac{\delta y}{2}}\right)\delta x\,\delta z = \left[\tau_{yx}\big|_y + \frac{\partial}{\partial y}\tau_{yx}\big|_y\frac{\delta y}{2} - \left(\tau_{yx}\big|_y + \frac{\partial}{\partial y}\tau_{yx}\big|_y\left(-\frac{\delta y}{2}\right)\right)\right]\delta x\,\delta z$$

$$= \frac{\partial}{\partial y}\tau_{yx}\,\delta y\,\delta x\,\delta z\Big|_{x,y,z}$$

SIMILARLY

$$\left(\tau_{zx}\big|_{z+\frac{\delta z}{2}} - \tau_{zx}\big|_{z-\frac{\delta z}{2}}\right)\delta x\,\delta y = \frac{\partial}{\partial z}\tau_{zx}\,\delta z\,\delta x\,\delta y\Big|_{x,y,z}$$

THUS, NET SURFACE FORCE IN X-DIRECTION IS

$$\boxed{\delta F_{sx} = \left(\frac{\partial}{\partial x}\sigma_{xx} + \frac{\partial}{\partial y}\tau_{yx} + \frac{\partial}{\partial z}\tau_{zx}\right)\delta x\,\delta y\,\delta z} \qquad (3)$$

NOW FROM (1)

$$\boxed{a_x = \frac{Du}{Dt} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t}} \quad (4)$$

FINALLY USING (3) AND (4) IN (2) AND CANCELLING THE COMMON TERM $\delta x \delta y \delta z$, we obtain the DIFFERENTIAL MOMENTUM EQN IN X-DIRECTION:

(NET SURFACE FORCE IN X-DIR)

$$\boxed{\rho\underbrace{\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right)}_{\text{(t.r.o.c. of ptcl. x-momentum)}} = \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + \underbrace{\rho g_x}_{\text{(BODY FORCE IN X-DIR)}}} \quad (5)$$

REPEATING THESE STEPS IN Y AND Z DIRECTIONS YIELDS Y AND Z MOMENTUM EQNS:

$$\boxed{\rho\frac{Dv}{Dt} = \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{zy}}{\partial z} + \rho g_y} \quad (6) \text{ y-MOMENTUM EQN}$$

$$\boxed{\rho\frac{Dw}{Dt} = \frac{\partial \sigma_{zz}}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \rho g_z} \quad (7) \text{ z-MOMENTUM EQN}$$

NOTE

$$\frac{Dv}{Dt} = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}$$

$$\frac{Dw}{Dt} = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}$$

NOTE THAT EQNS (5), (6) AND (7) ARE COMPLETELY GENERAL AND APPLY NOT ONLY TO FLUIDS, BUT ALSO TO SOLIDS (i.e., ELASTIC SOLIDS, PLASTIC SOLIDS) AND TO PLASMAS (IN THIS CASE THE BODY FORCE IS DUE TO ELECTROMAGNETIC INTERACTIONS BETWEEN CHARGED PTCLS.).

NOW, WE WANT TO EXPRESS τ's AND σ's IN TERMS OF VELOCITIES AND PRESSURE -- THIS WILL LEAD TO 4 EQNS (3 MOMENTUM EQNS IN (5), (6) & (7) PLUS THE CONTINUITY EQN) IN 4 UNKNOWNS, u, v, w AND P. STATED W/OUT PROOF WE HAVE THE FOLLOWING (WHICH ARE GENERALIZATIONS OF NEWTON'S LAW OF VISCOSITY):

$$\boxed{\begin{aligned} \sigma_{xx} &= -p + 2\mu \frac{\partial u}{\partial x} \\ \sigma_{yy} &= -p + 2\mu \frac{\partial v}{\partial y} \\ \sigma_{zz} &= -p + 2\mu \frac{\partial w}{\partial z} \\ \tau_{yx} = \tau_{xy} &= \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \\ \tau_{zx} = \tau_{xz} &= \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \\ \tau_{zy} = \tau_{yz} &= \mu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) \end{aligned}}$$

CONSTITUTIVE RELNS. FOR NEWTONIAN FLUID (8)

INSERTING (8) INTO (6) (7) & (8) LEADS TO THE FINAL MOMENTUM EQNS (WHICH APPLY ONLY TO NEWTONIAN FLUIDS W/ CONSTANT VISCOSITY);

$$\underset{\text{I}}{\rho \frac{Du}{Dt}} = \underset{\text{II}}{-\frac{\partial P}{\partial x}} + \underset{\text{III}}{\mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)} + \underset{\text{IV}}{\rho g_x} \qquad (9A)$$

$$\rho \frac{Dv}{Dt} = -\frac{\partial P}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) + \rho g_y \qquad (9B)$$

$$\rho \frac{Dw}{Dt} = -\frac{\partial P}{\partial z} + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) + \rho g_z \qquad (9C)$$

PHYSICAL INTERPRETATION:

I = T.R.O.C. OF MOMENTUM/VOL

II = NET PRESS. FORCE/VOL.

III = NET VISCOUS (FRICTION) FORCE/VOL.

IV = BODY FORCE/VOL

CONS. OF MASS (INCOMPRESSIBLE FLOW

$$\boxed{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0} \qquad (10)$$

SOLN. OF NAVIER-STOKES EQNS

① UNDERSTAND PHYSICS OF FLOW

② MAKE ASSUMPTIONS / SIMPLIFICATIONS REGARDING FLOW.

COMMON QUESTIONS:

A) IS FLOW STEADY? – YES → $\frac{\partial}{\partial t} \neq 0$
NO → $\frac{\partial}{\partial t} = 0$

B) IS FLOW INCOMPRESSIBLE? ($Ma \lesssim 0.3$)

C) IS FLOW 1, 2, OR 3-DIMENSIONAL? (i.e., HOW MANY VELOCITY COMPONENTS ARE NEEDED TO ACCURATELY DESCRIBE FLOW)

D) IS FLOW INTERNAL? (eg., PIPE FLOW)

a) IS FLOW FULLY-DEVELOPED? (BOUNDARY LAYER HAS GROWN TO CENTER OF FLOW AND VELOCITY DISTRIBUTION DOESN'T CHANGE IN FLOW DIRECTION)

E) IS FLOW LAMINAR OR TURBULENT?

a) MOST ENGINEERING FLOWS ARE TURBULENT – NUMERICAL OR EXPERIMENTAL SOLN. REQD.

b) MAY BE ABLE TO LOOK UP TRANSITION REYNOLDS NUMBER IF YOUR FLOW GEOMETRY IS SIMPLE

F) IS ONE COORDINATE SYSTEM MORE APPROPRIATE THAN ANOTHER?

a) SHAPE OF FLOW BOUNDARY DETERMINES APPROPRIATE COORD. SYSTEM; e.g., FLOW IN A CIRCULAR PIPE IS MOST EASILY SOLVED USING POLAR-CYLINDRICAL COORDS.) FLOW IN A RECTANGULAR DUCT IS BEST DETERMINED USING CARTESIAN COORDS.

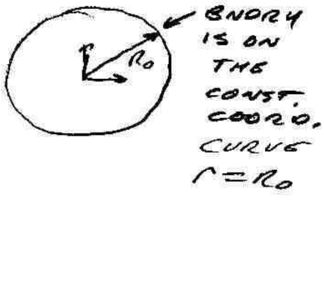
G) IS FLUID NEWTONIAN? IF SO, STRESS AND VELOCITY ARE RELATED AS GIVEN BY EQS. (8). IF NOT, THEN OTHER CONSTITUTIVE RELNS. MUST BE USED IN (5), (6) AND (7). [AIR & WATER, LIQUID METALS AND MANY SMALL CHAIN HYDROCARBON LIQUIDS ARE NEWTONIAN. EXAMPLES OF NON-NEWTONIAN FLUIDS INCL. BLOOD, PAINT, MOST POLYMER LIQUIDS.]

H) IS REYNOLDS NUMBER HIGH OR LOW? HIGH REYNOLDS NUMBER EXTERNAL FLOWS ARE TYPICALLY CHARACTERIZED

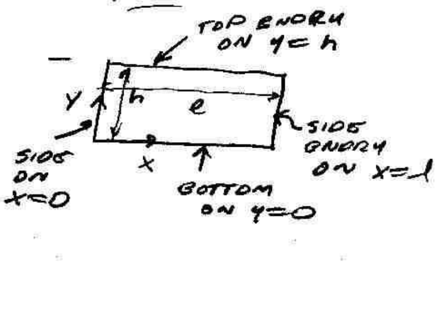
BY THIN BOUNDARY LAYERS NEAR SOLID (AND FLUID-FLUID BOUNDARIES) AND LARGE REGIONS OF INVISCID (AND POSSIBLY IRROTATIONAL ($\nabla \times \underline{u} = 0$) FLOW). INVISCID FLOW IS GOVERNED BY EULER EQNS (= NAV. STOKES EQNS W/ $\mu = 0$), WHILE INVISCID, IRROTATIONAL FLOW CAN BE SOLVED USING VELOCITY POTENTIALS. (SEE SEC. 6.4 FOR INTRODUCTION). [BOUNDARY LAYERS CAN BE SOLVED SEPARATELY — SEE CH. 9 FOR EXAMPLES.]

I) IS GEOMETRY 'SIMPLE' ENOUGH TO ALLOW AN ANALYTICAL SOLN., i.e., DO FLOW BNDRYS COINCIDE WITH CONSTANT COORDINATE LINES: EXAMPLES

PIPE FLOW



DUCT FLOW



a) IF GEOMETRY ISN'T SIMPLE AND YOU CAN'T MODEL THE ACTUAL GEOMETRY W/ A SIMPLE GEOMETRY, THEN A NUMERICAL (OR POSSIBLY VARIATIONAL) SOLN. IS REQUIRED.

J) OTHER QUESTIONS INCLUDE:

a) IS HEAT TRANSFER IMPORTANT? IF SO $\mu = \mu(T)$ AND $k = k(T)$ AND ENERGY EQN MUST ALSO BE SOLVED ($\rho c_p \frac{DT}{Dt} = \nabla(k \nabla T)$).

b) IS SURFACE TENSION IMPORTANT? TO ANSWER THIS RQRS. SCALING ARGUMENTS -- TAKE INTERMED. FLUIDS FOR MORE INFO.

c.) etc. etc.

3) ONCE AN INITIAL LIST OF ASSUMPTIONS / SIMPLIFICATIONS IS MADE, SIMPLIFY THE GOVERNING EQNS (i.e., 9A, 9B, 9C, & 10).

4) ATTEMPT TO SOLVE SIMPLIFIED EQNS., NOTE, ALL 4 EQNS. MUST

BE SATISFIED (BY YOUR SOLNS FOR, OR ASSUMPTIONS ON, u, v, w. AND P).

A) IF SOLN CAN'T BE OBTAINED, RETHINK ASSUMPTIONS AND SEE IF MORE SIMPLIFICATIONS CAN REASONABLY BE MADE. IF SO, TRY FOR ANOTHER SOLN.

B) IF SIMPLIFIED MODEL CAN'T BE SOLVED ANALYTICALLY, ATTEMPT SOLN. W/ A COMPUTATIONAL FLUIDS PACKAGE (WE HAVE A GOOD PKG. HERE – FLUENT)

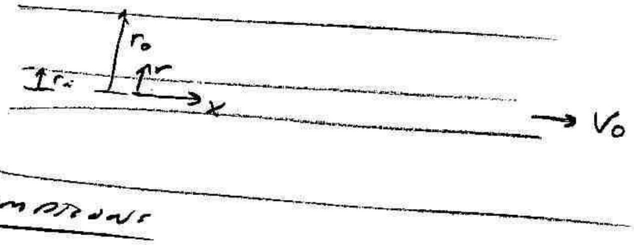
5) ONCE SOLN. OBTAINED, TEST IT:

a) FOR ANALYTICAL SOLN, TRY TO LOCATE A RELEVANT DATA (OR POSSIBLY PERFORM EXPERIMENTS YOURSELF).

b) ANALYTICAL SOLNS. CAN ALSO BE VALIDATED USING NUMERICAL SOLNS.

c) FOR NUMERICAL SOLNS., TEST YOUR SOFTWARE ON A SIMPLE RELATED FLOW FOR WHICH YOU HAVE KNOWN DATA.

EXAMPLE 6.93 – DETERMINE FOLLOWING LAMINAR, FULLY-DEV. FLOW FIELD.



ASSUMPTIONS

1) SINCE BOUNDARIES LIE ON CONSTANT radius curves, try cylindrical coord. eqs.
2) LAMINAR FLOW – GIVEN
3) INCOMPRESSIBLE $\Rightarrow \boxed{\rho = \text{CONST}}$
4) NO HEAT TRANSFER (ASSUME ISOTHERMAL CONDITIONS) $(\boxed{\mu = \text{CONSTANT}})$
5) FLOW IS GIVEN AS FULLY DEVELOPED $\Rightarrow \boxed{\frac{\partial}{\partial x} = 0}$
6) ASSUME NO SWIRL $\Rightarrow \boxed{V_\theta = 0}$
7) ASSUME NO RADIAL VELOCITY COMP. $\Rightarrow \boxed{V_r = 0}$
8) SINCE V_o IS CONSTANT, ASSUME STEADY FLOW $\Rightarrow \boxed{\frac{\partial}{\partial t} = 0}$
9) ASSUME HYDROSTATIC PRESS VARIATIONS NEGLIGIBLE $\Rightarrow \boxed{g =}$
10) ASSUME FLOW IS AXISYMMETRIC $\Rightarrow \boxed{\frac{\partial}{\partial \theta} = 0}$

START W/ THESE

SINCE WE'RE USING POLAR-CYLIND. COORDS. USE EQNS. 9.128a, b, c IN TEXT.

r-comp momentum:

$$\rho\left(\cancelto{0\,\frac{\partial}{\partial t}}{\frac{\partial v_r}{\partial t}} + \cancelto{0\,V_r=0}{v_r}\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\cancelto{V_r=0}{\frac{\partial v_r}{\partial \theta}} - \cancelto{V_\theta=0}{\frac{v_\theta^2}{r}} + v_z\cancelto{V_r=0}{\frac{\partial v_r}{\partial z}}\right) =$$

$$-\frac{\partial p}{\partial r} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\cancelto{0=V_r}{\frac{\partial v_r}{\partial r}}\right) - \cancelto{0=V_r}{\frac{v_r}{r^2}} + \frac{1}{r^2}\cancelto{0=V_r}{\frac{\partial^2 v_r}{\partial \theta^2}} - \frac{2}{r^2}\cancelto{0=V_\theta}{\frac{\partial v_\theta}{\partial \theta}} + \cancelto{0=}{\frac{\partial^2 v_r}{\partial z^2}}\right]$$

$$+ \cancelto{0}{\rho g_r}$$

$\Rightarrow$ r-comp $\Rightarrow$ $\boxed{\frac{\partial p}{\partial r} = 0}$ (E1)

θ-comp

$$\rho\left(\cancelto{V_\theta=0}{\frac{\partial v_\theta}{\partial t}} + v_r\cancelto{0=V_\theta}{\frac{\partial v_\theta}{\partial r}} + \frac{v_\theta}{r}\cancelto{0=V_\theta}{\frac{\partial v_\theta}{\partial \theta}} + \cancelto{0=V_\theta}{\frac{v_r v_\theta}{r}} + v_z\cancelto{0=V_\theta}{\frac{\partial v_\theta}{\partial z}}\right) =$$

$$-\frac{1}{r}\frac{\partial p}{\partial \theta} + \cancelto{0=g}{\rho g_\theta} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\cancelto{0=V_\theta}{\frac{\partial v_\theta}{\partial r}}\right) - \cancelto{0=V_\theta}{\frac{v_\theta}{r^2}} + \frac{1}{r^2}\cancelto{0=V_\theta}{\frac{\partial^2 v_\theta}{\partial \theta^2}} + \frac{2}{r^2}\cancelto{0=V_r}{\frac{\partial v_r}{\partial \theta}} + \cancelto{0=V_\theta}{\frac{\partial^2 v_\theta}{\partial z^2}}\right]$$

θ-comp $\Rightarrow$ $\boxed{\frac{\partial p}{\partial \theta} = 0}$ (E2)

Z-COMP MOM.

$$\rho\left(\cancel{\frac{\partial v_z}{\partial t}}^{0=\frac{\partial}{\partial t}} + \cancel{v_r}^{0=v_r}\frac{\partial v_z}{\partial r} + \cancel{\frac{v_\theta}{r}}^{0=v_\theta}\frac{\partial v_z}{\partial \theta} + v_z\cancel{\frac{\partial v_z}{\partial z}}^{0=\frac{\partial}{\partial z}}\right) = -\frac{\partial P}{\partial z} + \rho \cancel{g_z}^{0=g}$$

$$+\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{1}{r^2}\cancel{\frac{\partial^2 v_z}{\partial \theta^2}}^{0=\frac{\partial}{\partial\theta}=\frac{\partial^2}{\partial\theta^2}} + \cancel{\frac{\partial^2 v_z}{\partial z^2}}^{0=\frac{\partial}{\partial z}=\frac{\partial^2}{\partial z^2}}\right]$$

Z-COMP $\Rightarrow$ $$\boxed{-\frac{\partial P}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)\right] = 0} \quad (E$$

IN THIS EXAMPLE, FLOW IS DRIVEN BY THE MOTION OF THE CENTER ROD. THIS LEADS US TO ASK IF (OR POSSIBLY ASSUME THAT) THERE IS NO PRESSURE GRADIENT IN THE Z-DIRECTION (i.e., THE FLOW ISN'T PRESSURE-DRIVEN) WE CAN PROVE, USING (E3), (E1) & (E2) THAT

$\frac{\partial P}{\partial z}$ = CONSTANT = G. IN ORDER TO SET $\frac{\partial P}{\partial z}$ TO ZERO, WE WOULD HAVE TO BE TOLD THAT THERE'S NO PRESSURE GRADIENT. THUS, LETS OBTAIN THE MORE GEN'L SOLN WHERE $G = \frac{\partial P}{\partial z}$

ISN'T NECESSARILY ZERO.

TO SHOW THAT $\frac{\partial P}{\partial z}$ = CONSTANT,

$$\frac{\partial}{\partial z}(E3) \Rightarrow \frac{\partial^2 P}{\partial z^2} + \frac{\partial}{\partial z}\left(\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)\right]\right) =$$

$$\frac{\partial^2 P}{\partial z^2} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\left(\cancel{\frac{\partial v_z}{\partial z}}\right)\right)\right] = 0 \qquad 0 = \frac{\partial}{\partial z}$$

$$\Rightarrow \boxed{\frac{\partial^2 P}{\partial z^2} = 0} \qquad (E4)$$

NOW (E1) & (E2) SHOW THAT $P \neq fn(\theta)$ AND $P \neq fn(r)$.

THUS, (E4) IS AN <u>ORDINARY</u> 2^{nd} ORDER DERIVATIVE;

$$(E4) \Rightarrow \boxed{\frac{d^2 P}{dz^2} = 0} \qquad (E5)$$

NOW INTEGRATE (E5) ONCE TO GET

$$\frac{dP}{dz} = C_0 \qquad (E6)$$

AND LET $C_0 = G$.

INSERTING (E6) INTO (E3) YIELDS THE FINAL GOVERNING EQN FOR v_z;

$$\boxed{\frac{1}{r}\frac{d}{dr}\left(r\frac{dV_z}{dr}\right) = \frac{G}{\mu}} \quad (E7)$$

NOTE, SINCE $\frac{\partial}{\partial z} = 0$ AND $\frac{\partial}{\partial \theta} = 0$ (SEE ASSUMPTIONS), THEN

$V_z \neq fn(z)$ AND $V_z \neq fn(\theta)$.

THUS, $\boxed{V_z = V_z(r)}$ AND PARTIALS WITH RESPECT TO r BECOME ORDINARY DERIVS.

NOW INTEGRATE (E7):

$$\frac{d}{dr}\left(r\frac{dV_z}{dr}\right) = \frac{G}{\mu} r$$

$$\Rightarrow \int d\left(r\frac{dV_z}{dr}\right) = \int \frac{G}{\mu} r\,dr$$

$$\Rightarrow \quad r\frac{dV_z}{dr} = \frac{G}{\mu(2)} r^2 + c_1 \quad \text{INTEG. CONST.}$$

INTEGRATE AGAIN,

$$\frac{dV_z}{dr} = \frac{G}{2\mu} r + \frac{c_1}{r}$$

$$\Rightarrow \quad \boxed{V_z(r) = \frac{G}{4\mu} r^2 + c_1 \ln r + c_2} \quad (E8)$$

IN ORDER TO DETERMINE CONSTANTS c_1 & c_2, DEFINE BOUNDARY CONDITIONS ON v_z:

$$\boxed{v_z = 0 \quad \text{AT} \quad r = r_0} \quad (E9)$$

$$\boxed{v_z = V_0 \quad \text{AT} \quad r = r_i} \quad (E10)$$

USING (E9) IN (E8) $\Rightarrow$

$$v_z(r_0) = 0 = \frac{G}{4\mu} r_0^2 + c_1 \ln r_0 + c_2 \quad (E11)$$

(E10) IN (E8) $\Rightarrow$

$$v_z(r_i) = V_0 = \frac{G}{4\mu} r_i^2 + c_1 \ln r_i + c_2 \quad (E12)$$

SOLVE (E11) & (E12) FOR c_1 & c_2;

$$(E11) - (E12) \Rightarrow \quad c_1 \ln \frac{r_0}{r_i} = -V_0 + \frac{G}{4\mu}(r_0^2 - r_i^2)$$

$$\Rightarrow \quad (E13) \quad \boxed{c_1 = \left(V_0 - \frac{G}{4\mu}(r_0^2 - r_i^2)\right) \Big/ \ln\left(\frac{r_i}{r_0}\right)}$$

$$(E11) \Rightarrow \quad \boxed{c_2 = -\frac{G}{4\mu} r_0^2 - \left[V_0 - \frac{G}{4\mu}(r_0^2 - r_i^2)\right] \frac{\ln r_0}{\ln\left(\frac{r_i}{r_0}\right)}} \quad (E14)$$

SOLN $\Rightarrow$ USE (E13) & (E14) IN (E8)