LINEAR AND NONLINEAR HYDRODYNAMIC STABILITY

KEY IDEAS:

1) USE OF NORMAL MODES TO DESCRIBE/PREDICT LINEAR STABILITY

2) DERIVATION OF AMPLITUDE EQN FOR NONLINEAR EVOLUTION OF PERTURBATIONS TO THE BASIC STATE.

3) TEST FOR STABILITY OF BASIC STATE TO SMALL DISTURBANCES (PERTURBATIONS) - WE'LL USE BOTH LINEAR AND NONLINEAR STABILITY ANALYSES.

4) PREDICTION / DETERMINATION OF WEAKLY NONLINEAR SOLNS. WHEN A STABILITY PARAMETER R IS SLIGHTLY GREATER THAN ITS CRITICAL VALUE, R_c. R_c IS THE VALUE OF R AT WHICH LINEAR STABILITY ANALYSIS PREDICTS AN UNSTABLE BASIC STATE

I) FUNDAMENTAL CONCEPTS — HYDRODYNAMIC STABILITY

A) OBJECT OF ANY STABILITY ANALYSIS —

DETERMINE IF A GIVEN SOLN. ($\equiv$ BASIC FLOW), $\underline{u}(\underline{x}, t)$ [AND IN HEAT TRANSFER PROBLEMS, $T(\underline{x}, t)$], CAN ACTUALLY BE OBSERVED IN NATURE. THE APPROACH IS TO INTRODUCE A TYPICALLY SMALL DISTURBANCE AND DETERMINE IF THE DISTURBED FLOW: 1) RETURNS TO THE BASIC STATE, 2) TURNS INTO ANOTHER LAMINAR FLOW, OR 3) BECOMES TURBULENT. DISTURBANCES ($\equiv$ PERTURBATIONS) OF THESE KIND ARE REFERRED TO AS ASYMPTOTICALLY STABLE (OR SIMPLY STABLE), NEUTRALLY STABLE, OR UNSTABLE, RESPECTIVELY.

B) METHOD OF NORMAL MODES

⊕ USED TO ANALYZE STABILTY OF A LINEARIZED SYSTEM OF GOVERNING EQNS.

⊕ LINEARIZATION IS PERFORMED BY SIMPLY NEGLECTING PRODUCTS OF DIFFERENCES BETWEEN THE PERTURBED FLOW, $\underline{u}(\underline{x},t)$, AND THE BASIC FLOW, $\underline{U}(\underline{x},t)$. THUS, LETTING

$$\underline{u}'(\underline{x},t) = \underline{u}(\underline{x},t) - \underline{U}(\underline{x},t)$$

THEN TERMS LIKE

$$\underline{u}' \cdot \nabla \underline{u}'$$

IN THE MOMENTUM EQN. ARE NEGLECTED. SIMILARLY, WE NEGLECT SUCH TERMS IN NONLINEAR B.C.'S (e.g., THE KINEMATIC CONDITION AT FLUID-FLUID INTERFACES).

⊕ A BASIC FLOW (OR STATE) IS STABLE IF EVERY NORMAL MODE OF A COMPLETE SET OF NORMAL MODES DIES OUT IN TIME; A BASIC FLOW IS UNSTABLE IF ONE OR MORE NORMAL MODES GROW W/OUT BOUND IN TIME.

⊕ DUE TO THE SYSTEM'S <u>LINEARITY</u> (OBTAINED BY LINEARIZATION), WE CAN FOCUS ATTENTION ON A SINGLE, ARBITRARY NORMAL MODE AND DETERMINE THE SYSTEM'S STABILITY BASED ON THE BEHAVIOR OF THIS SINGLE NORMAL MODE.

⊕ A NORMAL MODE HAS THE TYPICAL FORM

$$\boxed{\underline{u}'(\underline{x},t) = \hat{u}(\underline{x},s,\underline{k})e^{st}e^{i\underline{k}\cdot\underline{x}}} \quad (1)$$

WHERE

$$\underline{u}'(\underline{x},t) = \underline{u}(\underline{x},t) - \underline{U}(\underline{x},t)$$

= DISTURBANCE VELOCITY FIELD

$\hat{u}(\underline{x},s,\underline{k})$ IS A FN. OF POSITION, s AND $\underline{k}$ DETERMINED BY INITIAL AND BOUNDARY CONDITIONS

$\underline{k}$ = WAVE NUMBER VECTOR FOR NORMAL MODE (= $k_x\hat{e}_1 + k_y\hat{e}_2 + k_z\hat{e}_3$ IN CARTESIAN PROBLEMS)

$\boxed{s = \sigma + i\omega}$ = COMPLEX GROWTH RATE FOR NORMAL MODE

s ARISES AS AN <u>EIGENVALUE</u>

IN THE <u>INITIAL-BOUNDARY VALUE PROBLEM</u> GOVERNING $\underline{u}'(\underline{x},t)$. s IS GENERALLY A FN. OF AT LEAST ONE <u>STABILITY PARAMETER</u> [e.g., THE REYNOLDS NUMBER IN PIPE FLOW STABILITY PROBLEMS, THE RAYLEIGH NUMBER IN THERMAL STABILITY PROBLEMS]

4. THE VALUE OF THE <u>REAL PART OF</u> s, σ, DETERMINES THE <u>STABILITY</u> OF THE <u>BASIC FLOW</u>:

$\sigma < 0 \Rightarrow$ BASIC FLOW IS STABLE TO THE MODE, OR EQUIVALENTLY, MODE IS <u>STABLE</u>

$\sigma > 0 \Rightarrow$ MODE IS <u>UNSTABLE</u>, i.e., BASIC FLOW IS <u>UNSTABLE</u>

$\sigma = 0 \Rightarrow$ MODE IS <u>NEUTRALLY STABLE</u>, i.e., BASIC FLOW WILL TRANSFORM TO ANOTHER STABLE FLOW (WHICH WILL BE OSCILLITORY IF $\omega \neq 0$).

- MARGINAL STABILITY OCCURS WHEN $\sigma = 0$, BUT WHERE σ BECOMES POSITIVE FOR SLIGHT CHANGES IN A STABILITY PARAMETER; A TYPICAL OBJECTIVE OF STABILITY ANALYSES IS TO DETERMINE THE CURVE OR SURFACE OF MARGINAL STABILITY, i.e., AN EQN. INVOLVING THE PARAMETERS DEFINING $\sigma = 0$. KNOWING THIS EQN. IS EQUIVALENT TO DETERMINING THE COMPLETE INSTABILITY CHARACTERISTICS OF THE BASIC FLOW.

  - IF $\omega \neq 0$ AS $\sigma \to 0^+$ THEN OSCILLATORY INSTABILITY OCCURS (NOTE $\sigma > 0$ IMPLIES SLOW GROWTH OF DISTURBANCE)

  - IF $S = 0$ when $\sigma = 0$ (i.e., $\omega = \sigma = 0$), THEN THE PERTURBED BASIC STATE SETTLES DOWN TO A DIFFERENT (AND STABLE) BASIC STATE. THIS IS CALLED EXCHANGE OF STABILITIES.

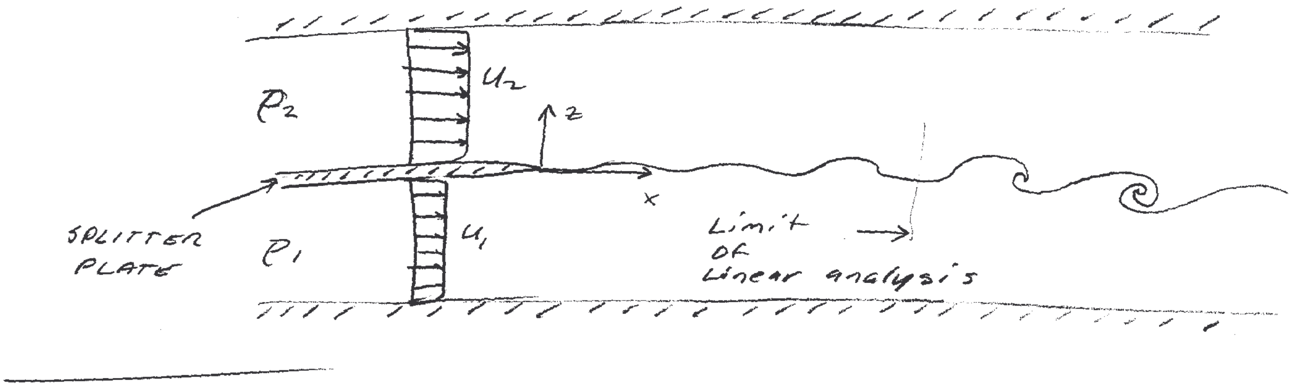
⊕ LINEAR STABILITY ANALYSES ASSUME THAT DISTURBANCES TO THE BASIC FLOW ARE INFINITESIMAL; THIS ASSUMPTION UNDERLIES THE BASIC APPROACH OF LINEARIZING THE GOVERNING EQNS.

II) ILLUSTRATION - KELVIN-HELMHOLTZ INSTABILITY

BACKGROUND - WHEN TWO FLUIDS MOVE PAST ONE ANOTHER IN PLANAR FLOW AT SOME RELATIVE SPEED, THE INTERFACE BETWEEN THE TWO FLUIDS CAN ROLL UP INTO A SERIES OF CRESTS AND VALLEYS - SEE ATTACHED PHOTOS.

ANALYSIS - WE FOLLOW KELVIN WHO IDEALIZED THE FLOW AS TWO IRROTATIONAL FLOWS SEPARATED BY AN INFINITESIMALLY THIN VORTEX SHEET. THE VORTEX SHEET MODELS THE THIN VISCOUS BOUNDARY LAYER THAT EXISTS BETWEEN THE TWO IRROTATIONAL FLOWS.

NOTE: ONE WAY TO EXPERIMENTALLY GENERATE THIS KIND OF FLOW IS AS FOLLOWS



BASIC FLOW

(E1) $\underline{U} = \begin{cases} U_2\, \hat{\imath} \\ U_1\, \hat{\imath} \end{cases} \qquad \rho = \begin{cases} \rho_2 \\ \rho_1 \end{cases} \qquad P = \begin{cases} P_0 - \rho_2 g z & z > 0 \\ P_0 - \rho_1 g z & z < 0 \end{cases}$

INTERFACE POSITION

(E2) $$z = F(x, y, t)$$

ASSUME THAT DISTURBED FLOWS IN REGIONS ① & ② REMAIN IRROTATIONAL. (PHYSICALLY, THIS MEANS WE ASSUME THAT VORTICITY W/IN BOUNDARY LAYERS AND W/IN THE VORTEX SHEET REMAINS CONFINED, i.e., OUTWARD DIFFUSION OF VORTICITY IS LIMITED BY STREAM-WISE ADVECTION.

THUS,

$$\underline{u}_1 = \underline{U}_1 + \underline{u}_1' = \nabla \phi_1 \qquad (E3)$$

$$\underline{u}_2 = \underline{U}_2 + \underline{u}_2' = \nabla \phi_2 \qquad (E4)$$

CONTINUITY GIVES <u>GOVERNING</u> <u>EQNS</u> FOR ϕ_1 AND ϕ_2:

$$\nabla \cdot \underline{u}_1 = \boxed{\nabla^2 \phi_1 = 0} \qquad (E5)$$

$$\nabla \cdot \underline{u}_2 = \boxed{\nabla^2 \phi_2 = 0} \qquad (E6)$$

BOUNDARY CONDITIONS

AS

$$\boxed{\begin{array}{ll} z \to -\infty & \phi_1 \to U_1 x \\ z \to \infty & \phi_2 \to U_2 x \end{array}} \qquad \begin{array}{r} (E7a) \\ (E7b) \end{array}$$

AT $z = F(x,y,t)$ $\quad P_1 = P_2$ $\qquad (E8)$

SINCE BERNOULLI'S EQN. APPLIES IN FLUID ① AND ② THEN

$$\rho_1 \phi_{1,t} + P_1 + \tfrac{1}{2}\rho_1 \nabla\phi_1 \cdot \nabla\phi_1 + \rho_1 g z = C_1(t) \qquad (E8a)$$

$$\rho_2 \phi_{2,t} + P_2 + \tfrac{1}{2}\rho_2 \nabla\phi_2 \cdot \nabla\phi_2 + \rho_2 g z = C_2(t) \qquad (E8b)$$

WHERE $C_1(t)$ AND $C_2(t)$ ARE TIME-DEPENDENT FNS. THAT ARISE WHEN INTEGRATING EULER'S EQN $\left(\rho \frac{D\underline{u}}{Dt} = -\nabla P - \rho g \nabla z\right)$ FROM

ARBITRARY REFERENCE POINTS IN FLUIDS ① & ② TO POSITION $\underline{x}$ IN EACH FLUID LAYER. USING (E8a) AND (E8b) IN B.C. (E8) GIVES:

$$\left. \begin{aligned} &C_1(t) - \rho_1 \phi_{1,t} - \tfrac{1}{2}\rho_1 \nabla\phi_1 \cdot \nabla\phi_1 - \rho_1 g z = \\ &C_2(t) - \rho_2 \phi_{2,t} - \tfrac{1}{2}\rho_2 \nabla\phi_2 \cdot \nabla\phi_2 - \rho_2 g z \end{aligned} \right|_{\boxed{z = F(x,y,t)}} \quad (E9)$$

EQNS. (E8a) AND (E8b) ALSO APPLY TO THE UNDISTURBED BASIC FLOW (IN WHICH

$$\phi_1 = U_1 x \quad \text{AND} \quad \phi_2 = U_2 x \text{)}.$$

THUS, C_1 AND C_2 ARE RELATED AS FOLLOWS

$$\boxed{C_1 - \tfrac{1}{2}\rho_1 U_1^2 = C_2 - \tfrac{1}{2}\rho_2 U_2^2} \quad (E10)$$

NOTE, $z = F(x,y,t) = 0$ WHEN THE FLOW IS UNDISTURBED. NOTE TOO THAT C_1 AND C_2 ARE EVALUATED AT POINTS WHERE $\phi_{,t} + \nabla\phi\cdot\nabla\phi/2 + \frac{p}{\rho} + gz$ REMAIN CONSTANT, i.e., FAR FROM INTERFACE.

THE PROBLEM GOVERNING EVOLUTION OF THE DISTURBED BASIC FLOW, GIVEN BY (E5), (E6), (E7a), (E7b) AND (E9) (+ (E12) AND (E13) BELOW) APPLIES TO $\underline{\text{ANY}}$ FINITE DISTURBANCE. IN ORDER TO CONTINUE THE ANALYSIS, WE NOW

LIMIT ATTENTION TO SMALL DISTURBANCES FOR WHICH INTERFACE SLOPES

$$\frac{\partial F}{\partial x} << 1 \quad , \quad \frac{\partial F}{\partial y} << 1$$

AND DISPLACEMENTS, $F(x,y,t)$, ARE SMALL. PRIOR TO LINEARIZING (E9), HOWEVER, LET'S IMPOSE THE CONDITION THAT PARTICLES INITIALLY LOCATED ON THE INTERFACE ($z = F$) REMAIN THERE, i.e., THE INTERFACE DOESN'T 'BREAK' (LIKE OCEAN WAVES) AND DOESN'T DEVELOP AIR POCKETS. THUS, THE <u>KINEMATIC CONDITION</u> IS GIVEN BY

$$\frac{dz}{dt} = \frac{dF}{dt} = \frac{\partial F}{\partial t} + \frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} \qquad \text{(E11)}$$

REWRITING (E11) IN FLUIDS ① AND ② AT $z = F(x,y,t)$, AND NOTING THAT $\frac{dz}{dt} = w = \phi_{,z}$, $\frac{dx}{dt} = u = \phi_{,x}$, AND $\frac{dy}{dt} = v = \phi_{,y}$, WE OBTAIN

$$\phi_{1,z} = F_{,t} + \phi_{1,x}F_{,x} + \phi_{1,y}F_{,y} \qquad z = F \qquad \text{(E12)}$$

$$\phi_{2,z} = F_{,t} + \phi_{2,x}F_{,x} + \phi_{2,y}F_{,y} \qquad z = F \qquad \text{(E13)}$$

NOW LET

$$\phi_1 = U_1 x + \phi_1' \qquad \text{(E14)}$$

$$\phi_2 = U_2 x + \phi_2' \qquad \text{(E15)}$$

NEGLECT PRODUCTS OF SMALL QTYS, AND TRANSFER B.C.'S ON $z=F$ TO $z=0$ TO OBTAIN:

$$\boxed{\begin{array}{ll} \nabla^2 \phi_1' = 0 & z<0 \\ \nabla^2 \phi_2' = 0 & z>0 \end{array}} \qquad \begin{array}{l}\text{(E16)} \\ \text{(E17)}\end{array}$$

$$\nabla \phi_1 = \underline{u}_1 \to U_1 \hat{\imath} \qquad z \to -\infty$$

$$\Rightarrow \boxed{\nabla \phi_1' \to 0 \qquad z \to -\infty}$$

ALSO

$$\boxed{\nabla \phi_2' \to 0 \qquad z \to \infty}$$

E18

(E19)

(E9) $\Rightarrow$

$$C_1(t) - \rho_1 \phi_{1,t}' - \tfrac{1}{2}\rho_1 [U_1\hat{\imath} + \nabla\phi_1'] \cdot [U_1\hat{\imath} + \nabla\phi_1'] - \rho_1 g F =$$

$$C_2(t) - \rho_2 \phi_{2,t}' - \tfrac{1}{2}\rho_2 [U_2\hat{\imath} + \nabla\phi_2'] \cdot [U_2\hat{\imath} + \nabla\phi_2'] - \rho_2 g F$$

USING (E10) AND NEGLECTING $\nabla\phi_1' \cdot \nabla\phi_1'$ AND $\nabla\phi_2' \cdot \nabla\phi_2'$, WE GET

$$\text{(E20)} \qquad \rho_1 \phi_{1,t}' + \rho_1 U_1 \phi_{1,x}' + \rho_1 g F = \rho_2 \phi_{2,t}' + \rho_2 U_2 \phi_{2,x}' + \rho_2 g F$$

$$\boxed{z=0}$$

SIMILARLY, (E12) AND (E13) BECOME

$$\boxed{\begin{array}{lll} \phi'_{1,z} = F_{1t} + U_1 F_{1x} & & z = 0 \\ \phi'_{2,z} = F_{1t} + U_2 F_{1x} & & z = 0 \end{array}} \qquad \text{(E21)}$$

NEXT, INTRODUCE <u>NORMAL</u> <u>MODES</u> BY ASSUMING SOLNS OF THE FORM

$$\begin{aligned} \phi_1' &= \hat{\phi}_1(z)\, e^{st}\, e^{i(kx+ly)} \\ \phi_2' &= \hat{\phi}_2(z)\, e^{st}\, e^{i(kx+ly)} \\ F &= \hat{F}\, e^{st}\, e^{i(kx+ly)} \end{aligned} \qquad \text{(E22)}$$

$$\text{(E16)} \Rightarrow \quad \hat{\phi}_{1,zz} - \hat{\phi}_1(k^2 + l^2) = 0$$

$$\Rightarrow \quad \hat{\phi}_1 = A_1 e^{-\tilde{k}z} + B_1 e^{\tilde{k}z}$$

WHERE $\quad \tilde{k} = \sqrt{k^2 + l^2}$ = wave number amplitude

$$\text{(E18)} \Rightarrow \quad A_1 = 0$$

$$\Rightarrow \quad \boxed{\hat{\phi}_1(z) = B_1 e^{\tilde{k}z}} \qquad \text{(E23)}$$

SIMILARLY

$$\boxed{\hat{\phi}_2(z) = B_2 e^{-\tilde{k}z}} \qquad \text{(E24)}$$

NOW USE (E20), (E21) TO OBTAIN 3 EQNS IN THE 3 UNKNOWNS, $\hat{F}$, B_1, AND B_2:

(E20) ⇒

$$\rho_1[sB_1 + U_1 iB_1 k + g\hat{F}] = \rho_2[sB_2 + U_2 iB_2 k + g\hat{F}] \qquad (E25)$$

(E21) ⇒

$$B_1\tilde{k} = s\hat{F} + U_1\hat{F}ik \qquad (E26)$$

$$-B_2\tilde{k} = s\hat{F} + U_2\hat{F}ik \qquad (E27)$$

(E26) ⇒

$$B_1 = \hat{F}(s + iU_1 k)/\tilde{k} \qquad (E28)$$

(E27) ⇒

$$B_2 = -\hat{F}(s + iU_2 k)/\tilde{k} \qquad (E29)$$

INSERTING (E28) & (E29) INTO (E25) YIELDS AN EIGENVALUE RELATION FOR s:

$$\rho_1\left[\frac{(s + iU_1 k)^2}{\tilde{k}} + g\right] = \rho_2\left[-\frac{(s + iU_2 k)^2}{\tilde{k}} + g\right]$$

OR

$$\boxed{\rho_1[(s + iU_1 k)^2 + g\tilde{k}] = \rho_2[-(s + iU_2 k)^2 + g\tilde{k}]} \qquad (E30)$$

SOLVE (E30) FOR S:

$$s^2(\rho_1+\rho_2) + s(i2k)(\rho_1 u_1 + \rho_2 u_2)$$

$$+ g\tilde{k}(\rho_1-\rho_2) - k^2(\rho_1 u_1^2 + \rho_2 u_2^2) = 0$$

$$\Rightarrow \quad s = -\frac{ik(\rho_1 u_1 + \rho_2 u_2)}{(\rho_1+\rho_2)} \pm$$

$$\pm \left[\frac{-\cancel{4}k^2(\rho_1 u_1 + \rho_2 u_2)^2}{\cancel{4}(\rho_1+\rho_2)^2} + \frac{-g\tilde{k}(\rho_1-\rho_2) + k^2(\rho_1 u_1^2 + \rho_2 u_2^2)}{(\rho_1+\rho_2)} \right]^{1/2}$$

$$= -\frac{ik(\rho_1 u_1 + \rho_2 u_2)}{(\rho_1+\rho_2)} \pm \left[\frac{k^2(-\cancel{\rho_1^2 u_1^2} - \cancel{\rho_2^2 u_2^2} - 2\rho_1\rho_2 u_1 u_2 + \cancel{\rho_1^2 u_1^2} + \cancel{\rho_2^2 u_2^2} + \rho_1\rho_2 u_1^2 + \rho_1\rho_2 u_2^2)}{(\rho_1+\rho_2)^2} - \frac{g\tilde{k}(\rho_1-\rho_2)}{(\rho_1+\rho_2)} \right]^{1/2}$$

$$\boxed{s = -\frac{ik(\rho_1 u_1 + \rho_2 u_2)}{(\rho_1+\rho_2)} \pm \left[\frac{k^2\rho_1\rho_2}{(\rho_1+\rho_2)^2}(u_1-u_2)^2 - \frac{g\tilde{k}(\rho_1-\rho_2)}{(\rho_1+\rho_2)} \right]^{1/2}} \quad * \quad (E31)$$

$$*$$

(E31) IS THE ALL-IMPORTANT EIGENVALUE RELATIONSHIP FOR THE GROWTH RATE S

NOTE (E31) PREDICTS TWO MODES OF GROWTH/INSTABILITY, CORRESPONDING TO THE $\pm$ SIGNS IN THE SECOND TERM.

SPECIAL CASES

⊙ NEUTRAL STABILITY $(\sigma \equiv Re(s) = 0)$:

IF $\boxed{\tilde{k} g(\rho_1^2 - \rho_2^2) \geq k^2 \rho_1 \rho_2 (U_1 - U_2)^2}$ THEN $Re(s) = 0$. IN THIS CASE, THE INTERFACE (WHICH IS INITIALLY FLAT IN THE BASIC STATE) SETTLES INTO PURE OSCILLATORY (WAVY) MOTION w/ $F(x,y,t) = \hat{F}_1 e^{i\omega_1 t} e^{i(kx+ly)} + \hat{F}_2 e^{i\omega_2 t} \cdot e^{i(kx+ly)}$, WHERE

$$\left.\begin{matrix}\omega_1\\ \omega_2\end{matrix}\right\} = -k\frac{(\rho_1 U_1 + \rho_2 U_2)}{(\rho_1+\rho_2)} \pm \left[-\frac{k^2\rho_1\rho_2(U_1-U_2)}{(\rho_1+\rho_2)^2} + g\frac{\tilde{k}(\rho_1-\rho_2)}{(\rho_1+\rho_2)}\right]$$

NOTE, k and l, the wave numbers in the x and y-directions, are parameters in the preceeding development. WE CHOOSE THESE AND THEN USE (E31) TO DETERMINE THE CORRESPONDING GROWTH RATE.

⊙ MARGINAL STABILITY ($\sigma \equiv Re(s) = 0$ + A SMALL CHANGE IN ANY PARAMETER $\rho_1, \rho_2, U_2, U_1, g$ CAUSES σ TO BECOME POSITIVE)

$$\boxed{\tilde{k} g(\rho_1^2 - \rho_2^2) = k^2 \rho_1 \rho_2 (U_1 - U_2)^2} \qquad (E32)$$

IN THIS CASE (REFERING TO (E31)),
ANY CHANGE IN ρ_1, ρ_2, U_1, U_2 and/or g which
CAUSES $K^2\rho_1\rho_2(U_1-U_2)^2$ TO BE LARGER THAN
$g\tilde{K}(\rho_1^2-\rho_2^2)$ WILL PRODUCE TWO MODES:
THE FIRST WILL BE (ASYMPTOTICALLY) STABLE
(CORRESPONDING TO THE "−" SIGN ON 2nd
TERM IN (E31)), WHILE THE SECOND WILL BE
UNSTABLE (CORRESPONDING TO "+" SIGN).
IN OTHER WORDS, THE FIRST WILL HAVE
$\sigma \equiv Re(s) < 0$ WHILE THE 2nd WILL HAVE
$\sigma > 0$. THUS, (E32) IMPLICITLY DEFINES
THE <u>SURFACE OF MARGINAL STABILITY</u>
FOR THE KELVIN-HELMHOLTZ PROBLEM.
IN SHORT, WE HAVE A <u>NORMAL MODE</u> THAT IS
<u>UNSTABLE</u> IF

$$\text{(E33)} \qquad \boxed{K^2\rho_1\rho_2(U_1-U_2)^2 > g\tilde{K}(\rho_1^2-\rho_2^2)} \quad \circledast$$

FOR SHORT WAVES, $K \gg 1$ and $\tilde{K} \gg 1$,
SO THAT

$$\frac{g\tilde{K}(\rho_1^2-\rho_2^2)}{K^2\rho_1\rho_2} = O(K^{-1}) \ll 1$$

THUS, THE INTERFACE IS <u>UNSTABLE</u> TO

SHORT WAVES IF $U_1 \neq U_2$.

SURFACE GRAVITY WAVES

IN THIS CASE, $U_1 = U_2 = \rho_2 = 0$. THUS,

(E31) GIVES

$$s = \pm(-\tilde{k}g)^{1/2} = \pm i\sqrt{\tilde{k}g}$$

$$\Rightarrow \sigma = Re(s) = 0$$

$$\omega = Im(s) = \pm\sqrt{\tilde{k}g}$$

$$\Rightarrow F(x,y,t) = \hat{F}e^{i(\omega t + kx + ly)}$$

THIS SHOWS THAT IN GENERAL SURFACE WAVES CAN TRAVEL IN AN OBLIQUE DIRECTION (WITH RESPECT TO THE X-AXIS). IF WE FIX AXES SO THAT WAVE TRAVELS IN X-DIRECTION THEN

$$F(x,t) = \hat{F}e^{i(\omega t + kx)}$$

THIS AND THE PROCEEDING EQN. DESCRIBE TRAVELING WAVES. (NOTE, WE HAVEN'T DIFFERENTIATED BETWEEN THE TWO WAVE FREQUENCIES, $\omega_1 = \sqrt{\tilde{k}g}$ AND $\omega_2 = -\sqrt{kg}$.)

THE WAVE SPEED, C, IS DETERMINED BY IMAGINING YOURSELF TRAVELING AT THE SAME SPEED AS SAY, THE WAVE CREST. THUS,

YOU'LL OBSERVE A WAVE OF FIXED SHAPE GIVEN BY $F = \hat{F} e^{i(\omega t + kx)} = \hat{F} e^{i\theta}$ WHERE THE PHASE θ IS CONSTANT.

HENCE

$$\theta = \omega t + kx$$

or $$x(t) = \frac{\theta}{k} - \frac{\omega}{k} t$$

$$\Rightarrow \boxed{C = \frac{dx}{dt} = -\frac{\omega}{k}}$$

THUS $$\boxed{|C| = \frac{\sqrt{kg}}{k} = \sqrt{\frac{g}{k}}}$$

RECALL THAT ~~THIS~~ IS THE PREDICTED WAVE SPEED FOR WAVES ON DEEP LAYERS (WHERE $\lambda/H \ll 1$ AND WHERE λ IS THE WAVELENGTH AND H IS THE LAYER DEPTH). [RECALL TOO THAT $\omega(k) = \sqrt{gk \tanh(kH)}$ IS THE DISPERSION RELATION WE DERIVED FOR gravity waves on a layer of liquid of depth H.]

$\Rightarrow$ THE LINEAR STABILITY THEORY IS CONSISTENT WITH THE THEORY OF SMALL AMPLITUDE SURFACE GRAVITY WAVES DERIVED EARLIER.

CONSISTENT WITH THE FACT THAT $\sigma = 0$ AND THAT SMALL AMPLITUDE SURFACE WAVES DO NOT GROW OR DECAY, THE LIN. STABILIT.

SHOWS THAT GRAVITY WAVES ARE <u>STABLE</u>. (ALTHOUGH THEY ARE OSCILLATORY THEY DO NOT GROW IN AMPLITUDE.)

⊕ <u>INTERNAL GRAVITY WAVES</u> – THESE CAN OCCUR AT THE INTERFACE BETWEEN TWO LIQUID LAYERS HAVING DIFFERENT DENSITIES, e.g., AT THE INLET TO ESTUARIES WHERE A FRESH WATER LAYER (~ 1 meter THICK) OVERLIES SALT WATER. THE SIMPLEST CASE IS WHERE $U_1 = U_2 = 0$. THUS, FROM (E31)

$$s = \pm\left[\frac{g\tilde{k}(\rho_2-\rho_1)}{(\rho_1+\rho_2)}\right]^{1/2} \qquad (E34)$$

IN THE CASE WHERE THE UPPER FLUID IS DENSER THAN THE LOWER ($\rho_2 > \rho_1$), THE INTERFACE IS ALWAYS UNSTABLE -- A HEAVIER FLUID ON TOP OF A LIGHTER ONE WILL TEND TO SINK INTO THE LIGHTER LAYER.

IN THE CASE WHERE THE LIGHTER FLUID IS ON TOP ($\rho_2 < \rho_1$), (e.g., AS IN ESTUARIES) (E34) SHOWS THAT $\sigma = Re(s) = 0$ AND $\omega = Im(s) = \pm\sqrt{\frac{g\tilde{k}(\rho_1-\rho_2)}{(\rho_1+\rho_2)}}$

SINCE $\sigma = 0$, THESE WAVES ARE <u>STABLE</u>.

THE CORRESPONDING PHASE VELOCITY IS

$$c = \frac{\omega}{\tilde{k}} = \sqrt{\frac{g(\rho_1 - \rho_2)}{\tilde{k}(\rho_1 + \rho_2)}} .$$

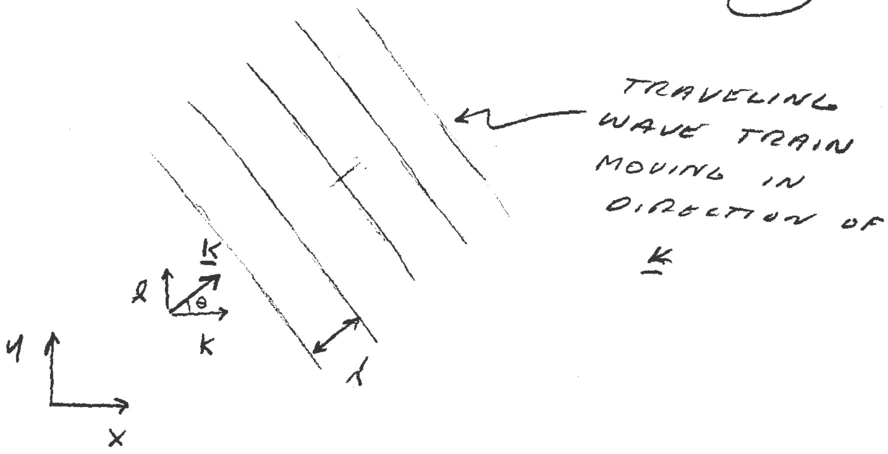
IN THE CASE OF FRESH WATER OVERLYING SALT WATER $\frac{(\rho_1 - \rho_2)}{\rho_1 + \rho_2} \approx 10^{-2}$. THUS, FOR EXAMPLE IF

$\tilde{k} \approx \frac{2\pi}{\lambda} \approx \frac{2\pi}{10^{-2} m}$, $c \approx \left(\frac{10}{2\pi}\right)^{1/2} \approx 1.3 m/s$.

ASIDE - WAVE VECTORS AND WAVE PROPAGATION

WE CAN REPRESENT A WAVE'S DIRECTION OF PROPAGATION BY ITS WAVE VECTOR

$\underline{K} = k\hat{e}_x + l\hat{e}_y$, WHERE $|\underline{K}| = \tilde{k} = \sqrt{k^2 + l^2} = \frac{2\pi}{\lambda}$ IS THE WAVE NUMBER AND λ IS THE THE WAVE LENGTH. k AND l ARE THE COMPONENTS OF $\tilde{K}$ IN THE x AND y-DIRECTIONS. A SIMPLE PICTURE IS USEFUL IN UNDERSTANDING HOW $\underline{K}$, k, l, AND THE DIRECTION OF PROPAGATION ARE RELATED



FROM PICTURE IT IS SEEN THAT THE DIRECTION OF PROPATION θ IS GIVEN BY

$$\theta = \tan^{-1} \frac{l}{k}$$

THE EXPRESSION FOR A TRAVELING WAVE MOVING IN THE VECTOR DIRECTION $\underline{x}$ IS

$$z = A \sin(\underline{k} \cdot \underline{x} - \omega t)$$
$$= A \sin(kx + ly - \omega t)$$

<u>END ASIDE</u>

④ **RAYLIEGH-TAYLOR INSTABILITY**

HERE $u_1 = u_2 = 0$ AND THE WHOLE SYSTEM IS GIVEN AN UPWARD or DOWNWARD VERTICAL ACCELERATION f. REPEATING THE DERIVATION OF (E31) w/ g REPLACED BY $g+f$, WE OBTAIN

$$s = \pm \left[-\frac{(g+f)(\rho_1 - \rho_2)\tilde{K}}{(\rho_1 + \rho_2)} \right]^{1/2} \qquad \text{(E35)}$$

NOTE $f < 0$ for DOWNWARD ACCELERATION AND $f > 0$ for UPWARD ACCELERATION.

THUS, IF $\rho_1 > \rho_2$ (HEAVY FLUID BELOW LIGHT) THEN BASIC STATE ($u_1 = u_2 = 0$) IS UNSTABLE IF (AND ONLY IF) $g + f < 0$, i.e. iff f is directed DOWNWARD WITH A MAGNITUDE GREATER THAN g ($|f| > |g|$). OTHERWISE, THE INTERFACE BETWEEN FLUIDS REMAINS UNDISTURBED AND THE BASIC STATE REMAINS STABLE. A SIMILAR ARGUMENT IN THE CASE WHERE HEAVY FLUID OVERLIES LIGHT FLUID ($\rho_1 < \rho_2$) IMPLIES THAT A STABLE INTERFACE AND BASIC STATE CAN BE MAINTAINED BY ACCELERATING THE SYSTEM DOWNWARD AT GREATER THAN 1-g.

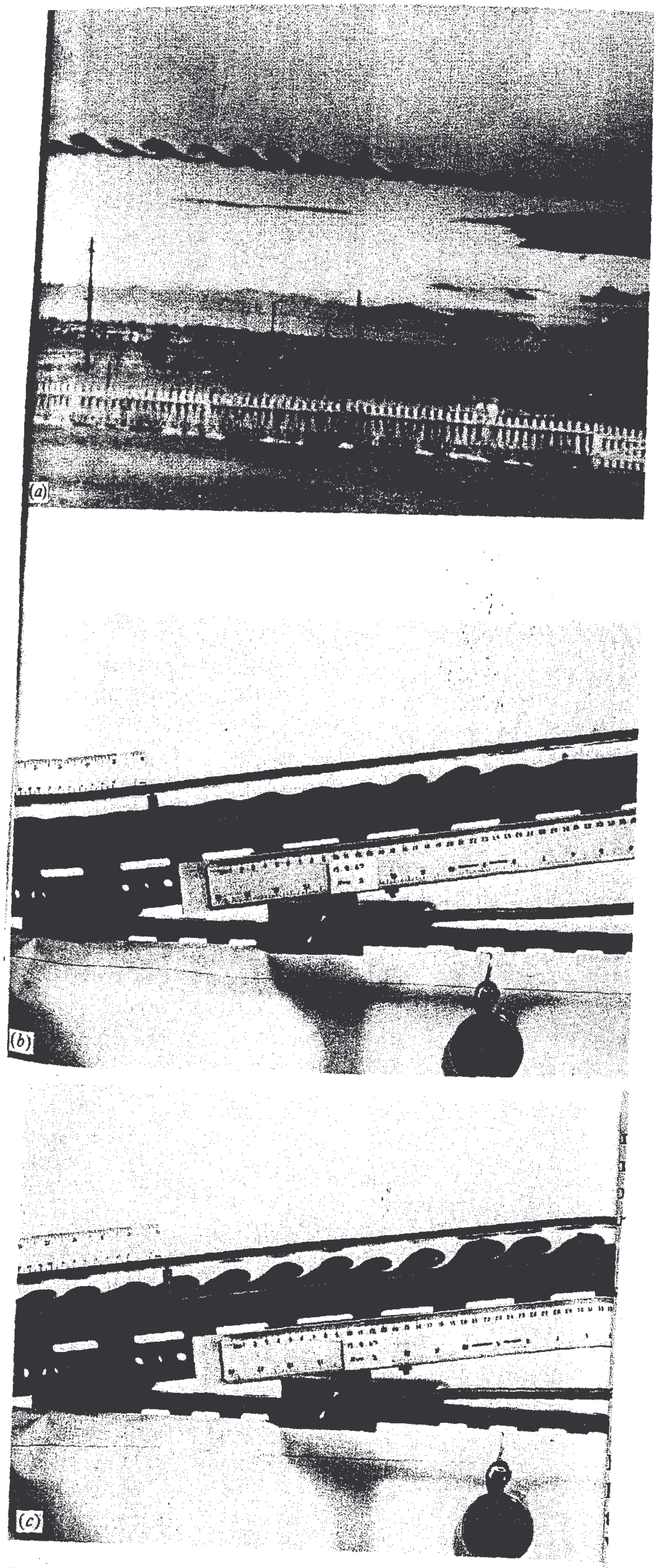


Fig. 1.4. Kelvin–Helmholtz instability. (*a*) Billow clouds near Denv Colorado, photographed by Paul E. Branstine. For the meteorologi details see Colson (1954). (*b*) Development of instability at the interface two fluids of equal depth in relative acceleration owing to the tilt of t channel. (From Thorpe 1968.) (*c*) The same run of Thorpe's experime about half a second later.

APPENDIX 1 – MARGINAL STABILITY ILLUSTRATED

FROM (E31), THE CONDITION FOR MARGINAL STABILITY IS

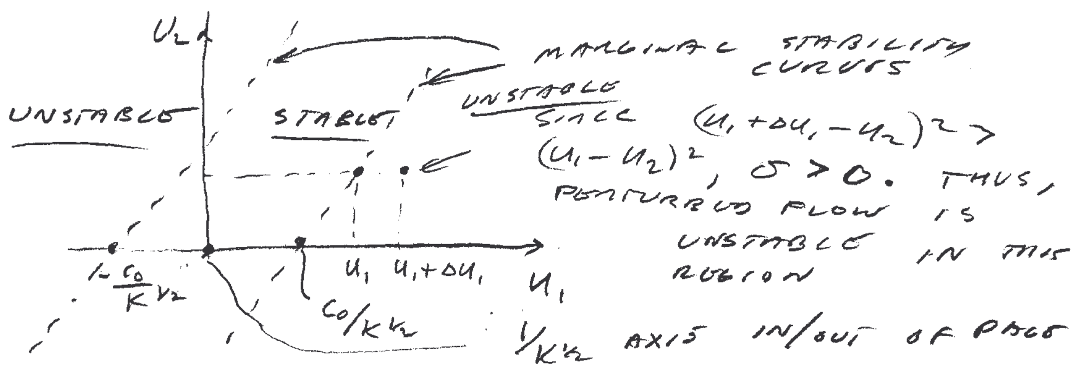
$$\sigma = 0 = k^2 \rho_1 \rho_2 (u_1 - u_2)^2 - g\tilde{k}(\rho_1^2 - \rho_2^2) \quad \text{(A.1)}$$

ASSUME THAT WE WANT TO DETERMINE STABILITY FOR A GIVEN TWO-FLUID SYSTEM (ρ_1, ρ_2 given) to NON-OBLIQUE DISTURBANCES ($l = \tilde{k} \cdot \hat{e}_2 = 0$; $\tilde{k} = k$). THUS, (A.1) BECOMES

$$(u_1 - u_2) = \pm \frac{c_0}{k^{1/2}}$$

$$\text{(A.2)} \qquad \Rightarrow \boxed{u_2 = u_1 \pm \frac{c_0}{k^{1/2}}} \qquad c_0^2 = \frac{g(\rho_1^2 - \rho_2^2)}{\rho_1 \rho_2} \qquad \boxed{\rho_1 > \rho_2}$$

THUS, GRAPHICALLY WE HAVE THE FOLLOWING STABILITY DIAGRAM, APPLICABLE TO ANY POSITIVE WAVENUMBER k



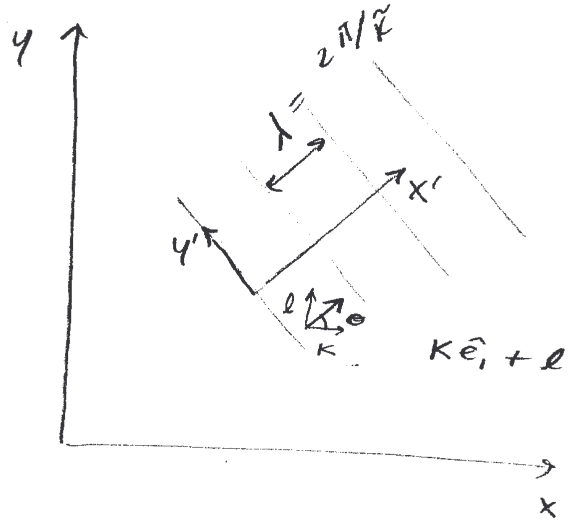
(DIAGRAM SHOWS $U_1 U_2$-PLANE AT GIVEN WAVENUMBER K.) (A2)

FROM DIAGRAM IT IS SEEN THAT WIDTH OF STABILITY REGION IS $\frac{2C_0}{K\gamma_2}$. THUS, FOR FIXED ρ_1, ρ_2 ($\rho_1 > \rho_2$), THIS REGION BECOMES INCREASINGLY NARROW (i.e., THE RANGE OF STABLE VELOCITIES BECOMES NARROWER) WITH INCREASING WAVE-NUMBER, i.e., w/ DECREASING WAVELENGTH. AS $K \to \infty$, $\frac{C_0}{K\gamma_2} \to 0 \Rightarrow$ FLOW IS UNSTABLE TO SMALL WAVE-LENGTH DISTURBANCES WHENEVER $U_1 \neq U_2$.

IN CASE WHERE $\rho_1 = \rho_2$, $C_0 = 0$ AND DISTURBED FLOW IS ONLY STABLE (MARGINALLY STABLE) WHEN $U_1 = U_2$. OTHERWISE, FLOW IS UNSTABLE (i.e., WHENEVER $U_1 \neq U_2$).

NOTE FROM (A.1) THAT THE BASIC FLOW IS ALWAYS STABLE TO TRANSVERSE DISTURBANCES, i.e., THOSE IN WHICH $K = 0$ AND $\tilde{K} = \ell$.

APPENDIX 2 – WAVENUMBER ILLUSTRATION



$$k\hat{e}_1 + l\hat{e}_2 = \underline{\tilde{k}}$$

$$\tilde{k} = |\underline{k}| = \sqrt{k^2 + l^2}$$

$$\tan\theta = l/k$$

$$\sin\theta = l/\tilde{k}$$

$$\cos\theta = k/\tilde{k}$$

A TRAVELING WAVE (WHICH PROPAGATES IN THE DIRECTION OF $\underline{\tilde{k}}$, i.e., IN X'-DIRECTION) IS DESCRIBED BY

$$e^{i(\tilde{k}x' - \omega t)}$$

NOW WE CAN SHOW THAT THIS IS IDENTICAL TO

$$e^{i(kx + ly - \omega t)}$$

BY NOTING THAT

$$x' = x\cos\theta + y\sin\theta$$

$$= x\frac{k}{\tilde{k}} + y\frac{l}{\tilde{k}}$$

THUS,

$$\boxed{\tilde{k}x' = kx + ly}$$