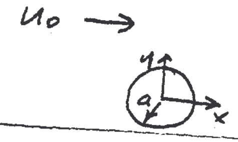
EARLY TIME

IMPULSIVE FLOW PAST A CIRCULAR CYLINDER – BOUNDARY LAYER SOLN.



HERE, FLUID AROUND A CYLINDER IS IMPULSIVELY ACCELERATED (OR THE CYLINDER ITSELF IS IMPULSIVELY ACCELERATED). IT IS OBSERVED THAT A THIN B. LAYER (A <u>RAYLEIGH</u> LAYER ANALOGOUS TO THAT PRODUCED BY A IMPULSIVELY ACCELERATED PLATE) SETS UP AROUND THE CYLINDER. HOWEVER, DUE TO ADVERSE PRESSURE GRADIENTS (WHICH WE'LL CALCULATE), THE B. LAYER SEPARATES. MODEL THE IMPULSIVE FLOW AND PRESS. FIELD OUTSIDE & INSIDE THE BOUNDARY LAYER.

OUTSIDE BOUNDARY LAYER, NAVIER STOKES EQNS ARE

$$\underline{V}_{,t} + \underline{V}\cdot\nabla\underline{V} = -\frac{\nabla p}{\rho} + \nu\nabla^2\underline{V} \qquad (1)$$

INTEGRATE (1) IN TIME OVER A SHORT INTERVAL BEFORE AND AFTER THE IMPULSE OCCURS

$$\underset{(I)}{\int_{-\epsilon}^{\epsilon} \underline{V}_{,t}\,dt} \quad \underset{(II)}{\int_{-\epsilon}^{\epsilon}(\underline{V}\cdot\nabla)\underline{V}\,dt} = \underset{(III)}{-\frac{1}{\rho}\int_{-\epsilon}^{\epsilon}\nabla p\,dt} + \underset{(IV)}{\int_{-\epsilon}^{\epsilon}\nu\nabla^2\underline{V}\,dt} \quad (2)$$

NOTE THAT THE TIME INTERVAL IS CHOSEN (ASSUMED) TO BE MUCH LONGER THAN THE CHARACTERISTIC ACOUSTIC TIME SCALE, $t_a = \frac{a}{c}$, WHERE a IS THE CYLINDER RADIUS AND c IS THE SPEED OF SOUND IN THE FLUID. SINCE TEMPORAL VARIATIONS IN THE PRESSURE FIELD OCCUR ON THE ACOUSTIC TIME SCALE (SINCE THESE VARIATIONS ARE TRANSMITTED ACOUSTICALLY) THEN WE'RE ASSUMING THAT P IS INDEPENDENT OF TIME, EXCEPT FOR A VERY SHORT TIME WHEN THE FLOW IS IMPULSIVELY STARTED. ALTHOUGH THE BOUND. LAYER FLOW IS TIME-DEPENDENT, AS IN MOST BOUND. LAYER PROBLEMS, THE B. LAYER FLOW DOES NOT AFFECT THE PRESSURE FIELD (AT LEAST AT A LEADING ORDER OF APPROXIMATION), RATHER THE P FIELD DETERMINES (FORCES) THE B. LAYER FLOW. OF COURSE, ONCE THE BOUNDARY LAYER SEPARATES, IT HAS A PROFOUND INFLUENCE ON THE ENTIRE FLOW (INCL. THE P FIELD) AND THEN DOES DETERMINE P.

NOW, LOOKING AT EACH TERM IN (2),

$(I) = \underline{v}(\underline{x}, \epsilon) = O(u_0)$ (u_0 = FREE STREAM SPEED FOLLOWING IMPULSIVE START-UP)

$(II) = O\left(\frac{u_0^2}{a}\epsilon\right)$

$(IV) = 0$ SINCE VORTICITY IS CREATED AT CYLINDER BNDRY $(r=a)$ and CONFINED TO BOUNDARY LAYER COATING THE CYLINDER

$$(III) = -\frac{1}{\rho}\int_{-\epsilon}^{\epsilon} \nabla P\,dt = -\frac{1}{\rho}\nabla\int_{-\epsilon}^{\epsilon} P\,dt = -\frac{1}{\rho}\nabla \underline{P} \qquad (3)$$

WHERE $\underline{P} = \int_{-\epsilon}^{\epsilon} P\,dt$ = IMPULSE PRESSURE

NOTE THAT AS BEFORE IN EARLIER EXAMPLES (HIGH Re. NO. FLOW, LOW Re. NO. FLOW, FREE SURFACE PROBLEMS, etc.), THE PRESSURE GRADIENT IS A "PASSIVE", OR MORE ACCURATELY, A "PRIMARY" DYNAMIC VARIABLE — IT IS GENERALLY DETERMINED BY (OR DETERMINES) THE DOMINANT FORCE (INERTIA, BODY, VISCOUS) IN ANY GIVEN PROBLEM. THUS, BASED ON THIS PHYSICAL FEATURE AND SCALING, WE KNOW THAT $\frac{1}{\rho}\int_{-\epsilon}^{\epsilon}\nabla P\,dt$ has to be of the same ORDER AS THE MOST

DOMINANT (TIME INTEGRATED) TERM --
IN THIS CASE $\underline{V}(\underline{x}, \epsilon)$. PHYSICALLY, THE IMPULSIVE FLOW IS DRIVEN BY A PRESSURE IMPULSE, FOR SHORT TIME SCALES, THE LOCAL PRESS. IMPULSE EQUALS THE LOCAL CHANGE IN PARTICLE MOMENTUM, $\underline{V}(\underline{x}, \epsilon)$. (per unit mass/($\rho_0$))
THE TIME SCALE ϵ MUST BE SMALL ENOUGH THAT $\int_{-\epsilon}^{\epsilon} \underline{V}\cdot\nabla\underline{V}\,dt$ remains NEGLIGIBLE COMPARED TO $\underline{V}(\underline{x}, \epsilon)$.
THUS

$$\frac{\int_{-\epsilon}^{\epsilon} \underline{V}\cdot\nabla\underline{V}\,dt}{\underline{V}(\underline{x}, t)} \approx \frac{\frac{u_0^2 \epsilon}{a}}{u_0} \approx \frac{u_0}{a}\epsilon << 1$$

$$\Rightarrow \boxed{\epsilon << a/u_0}$$

$$\Rightarrow \boxed{\frac{a}{c} << \epsilon << a/u_0} \quad (4)$$

For $a = 1\,cm$
$u_0 = 10\,m/s$
$c = 330\,m/s$
$3(10^{-5}) << \epsilon << 1(10^{-3})\,s$

(THE MODEL APPLIES OVER TIME SCALES SATISFYING (4))

IT TURNS OUT THAT B. LAYER SEPN. OCCURS AT $t \approx .35\, a/u_0$; thus, the CRITERION IN (4) IS NOMINALLY SATISFIED.

$\therefore$ (2) $\Rightarrow$

$$\underline{V}(x, \epsilon) = -\frac{1}{\rho} \nabla \underline{P} \qquad (5)$$

<u>CONTY</u>

$$\nabla \cdot \underline{U} = 0 \qquad (t = \epsilon) \qquad (6)$$

(5) + (6) $\Rightarrow$

$$\nabla^2 \underline{P} = 0 \qquad (t = \epsilon) \qquad (7)$$

<u>B.C.S</u>

AT $r = a + \delta = a(1 + \frac{\delta}{a}) = a + O(\frac{\delta}{a})$ THE radial velocity component is, to ORDER $\frac{\delta}{a}$, ZERO. PROOF: WITHIN BOUNDARY LAYER THE azimuthal velocity scale is

$$u_\theta \approx U_0$$

WHILE THE RADIAL SCALE IS (FROM CONTY)

$$u_r \approx \frac{U_0}{a}\delta.$$

THUS, the RADIAL NON-BOUNDARY LAYER velocity @ $r = a + \delta$ IS ON THE ORDER OF $\frac{U_0}{a}\delta$; FOR $\frac{\delta}{a} \ll 1$, $u_r(r = a + O(\frac{\delta}{a}), \theta) = 0 + \frac{\delta}{a} u_1$

$$= 0 + O(\frac{\delta}{a})$$

$$\Rightarrow \quad \boxed{\underline{P}_{,r} = u_r = 0} \text{ AT } r = a \quad (8)$$

(ACCURATE TO $O\left(\frac{\delta}{a}\right)$),

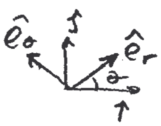
NOTE, THERE IS NO PHYSICAL CONDITION AT $r = a$ THAT CONSTRAINS THE AZIMUTHAL IMPULSE PRESSURE GRADIENT, $\underline{P}_{,\theta}$; THUS, THERE'S NO B.C. ON $\underline{P}_{,\theta}$ @ $r = a$.

[SINCE $\underline{P}$ IS ESSENTIALLY A velocity POTENTIAL (CONSISTENT W/ THE PHYSICAL FACT THAT VORTICITY IS ZERO OUTSIDE B. LAYER AND CONSISTENT W/ THE MATHEMATICAL RELATIONS IN (5)–(7)), THEN WE EXPECT THAT JUST AS WITH ANY velocity potential, A TANGENTIAL B.C. CANNOT BE IMPOSED ON $\underline{P}$ AT A SOLID BOUNDARY; i.e., WE CHOOSE TO SATISFY THE NO-PENETRATION CONDITION, $u_r = 0$, RATHER THAN THE NO SLIP CONDITION $u_\theta(r=a) = 0$.]

As $|\underline{x}| \to \infty$ $\quad \underline{u} \to U_0 \hat{\imath}$ $\quad$ (for $t > 0$).

$$\Rightarrow \underline{V}(\underline{x}, \epsilon) \to U_0 \hat{\imath} \qquad |\underline{x}| \to \infty$$

$$\Rightarrow -\nabla \underline{P} \to \rho U_0 \hat{\imath} \qquad |\underline{x}| \to \infty$$



$$\hat{\imath} = \hat{e}_r \cos\theta + \hat{e}_\theta \cos\left(\tfrac{\pi}{2} + \theta\right)$$

$$\hat{\imath} = \hat{e}_r \cos\theta - \hat{e}_\theta \sin\theta$$

$\therefore$

$$-\underline{P}_{,r} \to \rho U_0 \cos\theta \Rightarrow \underline{P}_1 = \rho U_0 \cos\theta\, r + g(\theta)$$

$$-\frac{1}{r}\underline{P}_{,\theta} \to \rho U_0 \sin\theta \Rightarrow \underline{P} = \rho U_0 \cos\theta\, r + f(r)$$

$g(\theta) = f(r) = \text{CONST}$

Let CONST $\equiv 0$ (NO LOSS OF generality, we just redefine $\underline{P}$ to be $\underline{P} \mp$ const)

ie $\quad \boxed{\underline{P}_1 \to \rho U_0 \cos\theta\, r \quad \text{AS} \quad r/a \to \infty} \qquad (9)$

B.V.P. FOR $\underline{P}$

$$\boxed{\begin{aligned} &\nabla^2 \underline{P} = 0 \\ &\underline{P}_{,r}(r = a, \theta; \epsilon) = 0 \\ &\underline{P}(r \to \infty, \theta; \epsilon) \to \rho U_0 \cos\theta\, r \end{aligned}} \qquad (10)$$

SOLVE (10) BY SEPN OF VBLS:

$$\underline{\rho} = R(r)F(\theta)$$

$$FR'' + \frac{R'}{r}F + \frac{RF''}{r^2} = 0$$

$$\Rightarrow \left(R'' + \frac{R'}{r}\right)\frac{r^2}{R} = -\frac{F''}{F} = \lambda^2 \leftarrow$$ sign on λ det'd by need for $\cos\theta$ in soln

$$\Rightarrow R'' + \frac{R'}{r} - \lambda^2\frac{R}{r^2} = 0$$ (EQUIDIMENSIONAL (EULER'S) EQN:)

SOLN $R(r) = Ar + \frac{B}{r}$

$$F'' + \lambda^2 F = 0$$

$$\Rightarrow F(\theta) = C\cos\lambda\theta + D\sin\lambda\theta$$

SINCE coeff. multiplying θ in FARFIELD B.C. IS 1, guess THAT $\boxed{\lambda = 1}$ (guess correct if all B.C.'s SATISFIED).

$$\Rightarrow \underline{\rho} = \left(Ar + \frac{B}{r}\right)(C\cos\theta + D\sin\theta)$$

SINCE THIS (TENTATIVE) SOLN FOR $\underline{\rho} \to Ar(C\cos\theta + D\sin\theta)$ AS $r/a \to \infty$

THEN SET $\boxed{D = 0}$

$$\Rightarrow \quad \underline{P} = \left(\tilde{A} r + \frac{\tilde{B}}{r}\right)\cos\theta \qquad \left(\tilde{A} = AC \quad \tilde{B} = BC\right)$$

$$\underline{P}_{,r}\Big|_{r=a} = 0 = \left(\tilde{A} - \frac{\tilde{B}}{a^2}\right)\cos\theta \quad \Rightarrow \quad \boxed{\tilde{A} = \frac{\tilde{B}}{a^2}}$$

$\underline{P} \to \rho U_0 (\cos\theta) r$ AS $r \to \infty$

OR

$$\tilde{A}\, r\cos\theta = \rho U_0 r \cos\theta$$

$$\Rightarrow \quad \boxed{\tilde{A} = \rho U_0}$$

$$\boxed{\tilde{B} = \rho a^2 U_0}$$

$$\therefore \quad \boxed{\underline{P}(r,\theta;\epsilon) = \rho U_0\left(r + \frac{a^2}{r}\right)\cos\theta} \qquad (11)$$

$$\boxed{\frac{a}{c} \ll \epsilon \ll \frac{a}{U_0}}$$

SOLN FOR PRESSURE IMPULSE FIELD (EARLY TIME)

BOUNDARY LAYER FLOW

$U = U_\theta$
$V = U_r$

CONTY

$$\frac{V}{r} + V_{,r} + \frac{1}{r}U_{,\theta} = 0$$

$$\mathcal{O}\left(\frac{U_s}{a}\right) \quad \mathcal{O}\left(\frac{U_s}{\sigma}\right) \quad \mathcal{O}\left(\frac{U_0}{a}\right)$$

SINCE $\frac{v}{r}/v_{1r} = O\left(\frac{\delta}{a}\right)$ neglect $\frac{v}{r}$ relative to v_{1r}. GET U_s BY BALANCING v_{1r} AGAINST $\frac{u_{1\theta}}{r}$:

$$\frac{U_s}{\delta} \approx \frac{U_0}{a}$$

$$\Rightarrow \boxed{U_s = U_0 \frac{\delta}{a}} \qquad (12)$$

(U_s = radial vel. scale in Bound. Lay.)

SIMPLIFIED CONT'Y:

$$\boxed{v_{1r} + \frac{u_{1\theta}}{r} = 0} \qquad (13)$$

<u>θ-momentum</u>

Determined by potential flow outside (soln in (11)).

Boundary layer

$$(14) \quad \underset{O\left(\frac{U_0}{t_s}\right)}{u_{1t}} + \underset{O\left(\frac{U_0^2 \delta}{\delta\, a}\right)}{v u_{1r}} + \underset{O\left(\frac{U_0^2}{a}\right)}{\frac{u u_{1\theta}}{r}} = -\frac{p_{1\theta}}{\rho r} + \nu\Big[\underset{O\left(\frac{U_0}{\delta^2}\right)}{u_{1rr}} + \underset{O\left(\frac{U_0}{\delta a}\right)}{\frac{u_{1r}}{r}} + \underset{O\left(\frac{U_0}{a^2}\right)}{\frac{u_{1\theta\theta}}{r^2}} \pm \underset{O\left(\frac{U_0 \delta}{a^3}\right)}{\frac{v}{r^2}}\Big]$$

FIND δ_s BY ARGUING THAT u_{1t} MUST BE OF SAME ORDER AS THE DOMINANT VISCOUS TERM, u_{1rr}. THIS ARGUMENT RECOGNIZES THAT THE PRESENT BOUNDARY

LAYER FLOW IS ANALOGOUS TO THE BOUND. LAYER ASSOCIATED w/ RAYLEIGH'S IMPULSIVELY ACCELERATED WALL PROBLEM. (11)

IN THAT PROBLEM, THE GOVERNING EQN IS GIVEN BY $u_t = \nu u_{yy}$. REPLACING EACH TERM BY IT'S SCALE, WE OBTAIN

$$\frac{\hat{u}_0}{\hat{t}_0} = \hat{\nu}\frac{\hat{u}_0}{\hat{\delta}^2} \Rightarrow \boxed{\hat{\delta}_s = \sqrt{\hat{\nu}\hat{t}_0}} \quad (15)$$

WHERE $\hat{t}_0$ IS THE TIME AT WHICH WE SEEK A SOLN. RECALL THAT ϵ IS THE TIME INCREMENT OVER WHICH

SOLN. (11) IS VALID; WE WILL

OBTAIN AN ASYMPTOTIC SOLN

IN terms of small dimless time

t_0, satisfying $t_0 << \epsilon << \frac{a}{u_0}$, WHERE $t_0 = \hat{t}_0/\hat{a}/\hat{u}_0$.

COMPARING VISCOUS TERMS IN (14),

WE SEE THAT $\frac{u_r/r}{u_{rr}} = O\left(\frac{\delta}{a}\right)$, $\frac{\frac{u_{\theta\theta}}{r^2}}{u_{rr}} = O\left[\left(\frac{\delta}{a}\right)^2\right]$,

AND $\frac{v/r^2}{u_{rr}} = O\left(\left(\frac{\delta}{a}\right)^3\right)$. THUS, WE COULD, AT THIS POINT, NEGLECT TERMS OF $O\left(\frac{\delta}{a}\right)$ OR SMALLER RELATIVE TO THE DOMINANT VISCOUS TERM

u_{rr} (i.e., neglect $\frac{u_r}{r}$; $\frac{u_{\theta\theta}}{r^2}$ AND $\frac{v}{r^2}$).

RATHER THAN DOING THIS, WE WILL USE A SYSTEMATIC APPROACH THAT WILL ALLOW US TO KEEP TRACK OF THE ERROR IN OUR SOLN.

IN PARTICULAR, WE WILL NONDIMENSIONALIZE THE θ-MOM. AND CONT'Y EQNS., ASSUME ASYMPTOTIC SOLNS IN TERMS OF THE DIMENSIONLESS TIME INCREMENT t_0 ($u = u_0 \oplus t_0 u_1 \oplus t_0^2 u_2 \ldots$), AND OBTAIN THE LEADING AND FIRST ORDER SOLNS, u_0, v_0, u_1, v_1 — IN DOING SO, WE WILL (INITIALLY) GENERATE A θ-MOM. EQN OF THE FORM

$$(15)\quad \boxed{\begin{aligned} 0 = {} & t^0[A] \oplus t_0[I] \oplus \epsilon_1[II] \oplus \epsilon_1^2[III] \oplus \epsilon_1^3[IV] \\ & \oplus \epsilon_1 t_0[V] \oplus \epsilon_1^2 t_0[VI] \oplus \epsilon_1^3 t_0[VII] \ldots \end{aligned}}$$

WHERE $\epsilon_1 = \delta/a$ AND WHERE THE TERMS INVOLVING ϵ_1, ϵ_1^2 AND ϵ_1^3 ([II], [III] & [IV]) CORRESPOND TO $\frac{u_{1r}}{r}$, $\frac{u_{1\theta\theta}}{r^2}$ AND $\frac{v}{r^2}$, RESPECTIVELY. THUS, IN OBTAINING AN ASYMPTOTIC SOLN. OF THE FORM

$$u = u_0 \oplus + u_1$$
$$v = v_0 \oplus + v_1$$

WE WILL BE IMPLICITELY ASSUMING THAT ALL TERMS BEYOND $O(t_0)$ IN EQN. (15) ARE MUCH SMALLER THAN $O(t_0)$. THIS WILL INTRODUCE THE FOLLOWING CONSTRAINT ON t_0: $\boxed{t_0 \gg \epsilon_1}$ (16)

SINCE $\hat{\delta} = O(\sqrt{\hat{\nu}\hat{t}_0})$ (to be shown) AND $t_0 = \hat{t}_0 / \hat{a}/\hat{u}_0$

(USING CARETS TEMPORARILY TO DISTINGUISH BETWEEN DIM'L AND NON-DIM'L QTYS.), THEN (LL) IS EQUIVALENT TO:

$$\frac{\hat{t}_0}{\hat{a}/\hat{u}_0} \gg \frac{\hat{\delta}}{\hat{a}} \approx \frac{\sqrt{\hat{\nu}\hat{t}_0}}{\hat{a}}$$

$$\Rightarrow \boxed{\hat{t}_0 \gg \frac{\hat{\nu}}{\hat{u}_0^2}}$$

EXAMPLE: FOR WATER $\hat{\nu} \approx 10^{-6}\, m^2/s$. SO FOR $\hat{u}_0 = 10\, m/s$ $\underline{\hat{t}_0 \gg 10^{-8} s}$ $\Rightarrow$ EASILY SATISFIED UNDER MOST CONDITIONS.

(RECALL THAT dim'l... $\hat{t}_0$ MUST ALSO BE MUCH SMALLER THAN ϵ, WHICH IN TURN MUST BE MUCH SMALLER THAN $\hat{a}/\hat{u}_0$; thus, $\hat{t}_0$ MUST ALSO SATISFY

$$\hat{t}_0 \ll \hat{a}/\hat{u}_0 .$$

FOR, $\hat{a} = 10^{-2} m$, $\hat{t}_0 \ll 10^{-3} s$.)

NON-DIM'LIZE θ-MOM., CONT'N:

(USE APPROACH WHERE SCALES ARE ACCOUNTED FOR IN THE GOV'ING EQNS RATHER THAN IN THE ASS'D ASYMPTOTIC SOLN.)

$\hat{t}_s = \hat{t}_0$, $\hat{u}_s = \hat{u}_0$, $\hat{v}_s = \hat{u}_0 \hat{\delta}/\hat{a} = \hat{u}_0 \epsilon_1$, $\hat{P}_s = \hat{\rho} u_0^2$

(NOTE $\hat{P}_s$ DET'D BY Pressure scale outside B.L.)

ALSO LET $y = \dfrac{\hat{r} - \hat{a}}{\hat{\delta}} \Rightarrow \hat{r} \equiv \hat{a} + \hat{\delta} y = \hat{a}(1 + \epsilon_1 y)$

$$\Rightarrow \frac{\partial}{\partial \hat{r}} = \frac{\partial}{\partial y}\frac{\partial y}{\partial \hat{r}} = \frac{1}{\hat{\delta}}\frac{\partial}{\partial y}$$

$$\frac{\partial^2}{\partial \hat{r}^2} = \frac{1}{\hat{\delta}^2}\frac{\partial}{\partial y^2}$$

$$\Rightarrow \quad u = \frac{\hat{u}}{\hat{u}_0}, \quad v = \frac{\hat{v}}{\hat{u}_0 \epsilon_1}, \quad t = \frac{\hat{t}}{\hat{t}_0}, \quad y = \frac{\hat{r} - \hat{a}}{\hat{\delta}},$$

$$P = \frac{\hat{P}}{\hat{\rho}\hat{u}_0^2}$$

θ-mom

(17)
$$\boxed{\begin{aligned} &\left(\frac{\hat{u}_0}{\hat{t}_0} u_{,t} + \frac{\hat{u}_0^2}{\hat{a}(1+\epsilon_1 y)} u u_{,\theta} + \frac{\hat{u}_0^2 \hat{\delta}}{\hat{\delta}\hat{a}} v u_{,y} = \right. \\ &-\frac{\hat{\rho}\hat{u}_0^2}{\hat{\rho}\hat{a}(1+\epsilon_1 y)} P_{,\theta} + \hat{\nu}\, u_0 \left[\!\left[\frac{1}{\hat{\delta}^2} u_{,yy} + \frac{1}{\hat{\delta}\,\hat{a}(1+\epsilon_1 y)} u_{,y} \right.\right. \\ &+ \frac{u_{,\theta\theta}}{\hat{a}^2(1+\epsilon_1 y)^2} \pm \left(\frac{\hat{\delta}}{\hat{a}}\right)\frac{1}{\hat{a}^2(1+\epsilon_1 y)^2} v^2 \left.\left.\right]\!\right] \end{aligned}}$$

DIVIDE (17) THROUGH BY $\frac{\hat{u}_o^2}{\hat{a}}$ AND EXPAND $\frac{1}{1+\epsilon_1 y}$ AS $1-\epsilon_1 y+\epsilon_1^2 y^2$ AND $\frac{1}{(1+\epsilon_1 y)^2}$ AS $1-2\epsilon_1 y+3\epsilon_1^2 y^2$ (TAYLOR EXPANSIONS OF $\frac{1}{1+x}$ AND $\frac{1}{(1+x)^2}$):

$$\boxed{\begin{aligned}&\frac{1}{\frac{\hat{u}_o \hat{t}_o}{\hat{a}}} u_{,t} + (1-\epsilon_1 y+\epsilon_1^2 y^2)\, u u_{,\theta} + v u_{,y} = \\ &-(1-\epsilon_1 y+\epsilon_1^2 y^2)\, p_{,\theta} + \frac{\hat{\nu}\hat{a}}{\hat{u}_o \hat{\delta}^2}\Big[\!\Big[u_{,yy} + \epsilon_1 (1-\epsilon_1 y+\epsilon_1^2 y^2)\, u_{,y} \\ &+ \epsilon_1^2 (1-2\epsilon_1 y+3\epsilon_1^2 y^2)\, u_{,\theta\theta} \pm \epsilon_1^3 (1-2\epsilon_1 y+3\epsilon_1^2 y^2)\, v^2 \Big]\!\Big]\end{aligned}} \tag{18}$$

NOW, NOTE THAT $\hat{\delta} \approx \sqrt{\hat{\nu}\hat{t}_o}$; AGAIN THIS IS PROVEN BY RECOGNIZING THAT W/IN THE TRANSIENT BOUNDARY LAYER, THE LOCAL PARTICLE ACCELERATION, $\hat{u}_{,\hat{t}}$, WILL BE COMPARABLE TO THE ADVECTING TERMS, $\frac{\hat{u}\hat{u}_{,\hat{\theta}}}{\hat{r}}$ AND $\hat{v}\hat{u}_{,\hat{r}}$, AND TO THE DOMINANT VISCOUS TERM, $\hat{\nu}\hat{u}_{,\hat{r}\hat{r}}$.

THUS,

$$\hat{u}_{,\hat{t}} \approx \hat{\nu}\hat{u}_{,\hat{r}\hat{r}}$$

or

$$\frac{\hat{u}_o}{\hat{t}_o} \approx \frac{\hat{\nu}\hat{u}_o}{\hat{\delta}^2} \Rightarrow \boxed{\hat{\delta} \approx \sqrt{\hat{\nu}\hat{t}_o}} \tag{19}$$

$\Rightarrow$ SET radial length scale δ EQUAL TO $\sqrt{\hat{\nu}\hat{t}_0}$.

THUS,

$$\boxed{\frac{\hat{\nu}\hat{a}}{\hat{u}_0\delta^2} = \frac{\hat{a}/\hat{u}_0}{\hat{t}_0} = \frac{1}{t_0}} \qquad (20)$$

AS MENTIONED, WE WILL SEEK AN LEADING AND 1st ORDER ASYMPTOTIC SOLN. IN TERMS OF t_0, WHERE t_0 IS SMALL. IN ADDITION WE WILL CONSTRAIN SOLNS TO CASES WHERE $t_0 \gg \epsilon_1$. THUS, WE WILL NEGLECT TERMS OF $O(\epsilon_1)$, $O(\epsilon_1^2)$, AND $O(\epsilon_1^3)$ IN EQN. (18) AND MULTIPLY THROUGH BY t:

$$(21) \qquad \boxed{u_{,t} \oplus t_0[u u_{,\theta} \oplus v u_{,y}] = \ominus t_0 P_{,\theta} \oplus u_{,yy} \;\Big|\; + O(\epsilon_1)}$$

SIGNIFIES THAT LARGEST NEGLECTED TERM IS OF $O(\epsilon_1)$.

B.C.'s:

$$(22) \qquad u(y=0,\theta,t) = 0$$

$$(23) \qquad v(y=0,\theta,t) = 0$$

$$(24) \qquad u(y\to\infty,\theta,t) = U(\theta)$$ = θ-vel. comp. outside B.LAYER

$$(25) \qquad v(y\to\infty,\theta,t) = V(\theta)$$ = r-vel. comp. outside B. LAYER

I.C.

$$(26) \qquad u(y,\theta,t=0) = U(\theta)$$

$$(27) \qquad v(y,\theta,t=0) = V(\theta)$$

NOTE THAT THE INITIAL CONDITIONS ARE ANALOGOUS TO THE INITIAL CONDITIONS EXTANT IN RAYLEIGH'S ACCELERATING WALL PROBLEM (MORE ACCURATELY, THEY ARE ANALOGOUS TO THOSE IN THE

PROBLEM WHERE FLOW ABOUT A PLANE WALL AT $y=0$ IS IMPULSIVELY ACCELERATED AT $t=0$).

NOTE TOO THAT $\bar{U}(\theta)$ & $\bar{V}(\theta)$ WILL BE DERIVED USING $\hat{\underline{V}}(\underline{x};\epsilon) = -\frac{1}{\rho}\hat{\nabla}\hat{\underline{p}}$ WHICH HOLDS OUTSIDE THE B. LAYER.

NOW, ASSUME AN ASYMPTOTIC SOLN;

(28) $$u(y,\theta,t) = u_0(y,\theta,t) \oplus t_0\, u_1(y,\theta,t)$$

(29) $$v(y,\theta,t) = v_0(y,\theta,t) \oplus t_0\, v_1(y,\theta,t)$$

$\mathcal{O}(1)$ problem (FOR u_0):

(30A) $$u_{0,t} = u_{0,yy}$$

(30B) $$u_0(y=0,t,\theta) = 0$$

(30C) $$u_0(y\to\infty,t,\theta) = \bar{U}(\theta)$$

(30D) $$u_0(y,t=0,\theta) = \bar{U}(\theta)$$

Let $\boxed{\tilde{u} = u_0 - \bar{U}(\theta)}$

(30A) → (30D) ⇒

(31A) $$\tilde{u}_{,t} = \tilde{u}_{,yy}$$

[NOTE, FOR $t \ll \epsilon/\bar{a}/\bar{u}_0$, $\bar{U} \neq fn(t)$]

(31B) $$\tilde{u}(y=0,t,\theta) = -\bar{U}(\theta)$$

(31C) $$\tilde{u}(y\to\infty,t,\theta) = 0$$

(31D) $$\tilde{u}(y,t=0,\theta) = 0$$

SOLVE BY LAPLACE TRANSFORM (OR SIMILARITY TRANSFORMATION):

$$\mathcal{L}(\tilde{u}_{,t}) = \int_0^\infty e^{-st}\tilde{u}_{,t}\,dt = -\tilde{u}(y,\theta,\cancel{t}=0)^{0} + s\mathcal{L}(\tilde{u})$$
$$= sf$$

where $f = \mathcal{L}(\tilde{u})$

$$\mathcal{L}(\tilde{u}_{,yy}) = f_{,yy}$$

$$\mathcal{L}(U(\theta)) = \int_0^\infty U(\theta)e^{-st}\,dt = \frac{U(\theta)}{s}$$

$$\Rightarrow \boxed{\begin{aligned} &f_{,yy} - sf = 0 \\ &f(y=0,\theta) = -\frac{U(\theta)}{s} \\ &f(y\to\infty,\theta) = 0 \end{aligned}} \begin{aligned} &\leftarrow \mathcal{L}(31A) \\ &\leftarrow \mathcal{L}(31B) \\ &\leftarrow \mathcal{L}(31C) \end{aligned}$$

$$f = ae^{-s^{1/2}y} + be^{s^{1/2}y}$$

$$f(y\to\infty,\theta) = 0 \Rightarrow \boxed{b=0}$$

$$f(y=0) = -\frac{U(\theta)}{s} = a$$

$$\Rightarrow \boxed{f(s,y,\theta) = -\frac{U(\theta)}{s}e^{-s^{1/2}y}}$$

FROM TABLE OF LAPLACE X-FORMS

$$\tilde{u} = \mathcal{L}^{-1}(f) = -U(\theta)\,\mathrm{erfc}\left(\frac{y}{2\sqrt{t}}\right)$$

$$\Rightarrow \quad u_0(y,\theta,t) = U(\theta) + \tilde{u} = U(\theta)\left[1 - \mathrm{erfc}\left(\frac{y}{2\sqrt{t}}\right)\right]$$

$$\boxed{u_0(y,\theta,t) = U(\theta)\,\mathrm{erf}\left(\frac{y}{2\sqrt{t}}\right)} \qquad (32)$$

GET V_0 FROM CONT'Y

$$y = \frac{\hat{r} - \hat{a}}{\delta}$$

$$\frac{1}{\hat{r}}\hat{u}_{,\theta} + \hat{v}_{,\hat{r}} = \frac{1}{\cancel{a}(1+\epsilon_1 y)}\cancel{\hat{u}_0}\,u_{,\theta} + \cancel{\left(\frac{\hat{u}_0 \delta}{\delta}\right)}\,V_{,y} = 0$$

$$\Rightarrow \boxed{u_{,\theta} + V_{,y} = 0} \qquad (33)$$

$$\Rightarrow \quad V_0 = -\int_0^y u_{0,\theta}\,dy$$

$$= -U'(\theta)\int_0^y \mathrm{erf}\left(\frac{y'}{2\sqrt{t}}\right)dy'$$

TABLE OF INTEGRALS

$$\boxed{V_0 = -2\frac{dU(\theta)}{d\theta}t^{1/2}\left[m\,\mathrm{erf}(m) - \frac{1}{\sqrt{\pi}}\left(1 - e^{-m^2}\right)\right], \quad m = \frac{y}{2\sqrt{t}}} \qquad (33)$$

GET $U(\theta)$;

$$\underline{\hat{V}}(\hat{x}, \epsilon) = -\frac{1}{\rho}\hat{\nabla}\hat{P}$$

$$= -\frac{1}{\cancel{\rho}}\left[\hat{e}_r\frac{\partial}{\partial\hat{r}} + \frac{\hat{e}_\theta}{r}\frac{\partial}{\partial\theta}\right]\cdot\left[\cancel{\hat{\rho}}\,\hat{u}_0\left(\hat{r} + \frac{\hat{a}^2}{\hat{r}}\right)\cos\theta\right]$$

$\hat{U}$ = θ-COMP OUTSIDE B. LAYER
$\hat{V}$ = r - " " " "

$$\boxed{\hat{U} = \frac{\hat{u}_0}{\hat{r}}\left(\hat{r} + \frac{\hat{a}^2}{\hat{r}}\right)\sin\theta} = \hat{U}(\theta,\hat{r}) \quad (34)$$

$$\boxed{\hat{V} = -\hat{u}_0\left(1 - \frac{\hat{a}^2}{\hat{r}^2}\right)\cos\theta} = \hat{V}(\theta,\hat{r}) \quad (35)$$

$$\boxed{U(\theta) = \frac{\hat{U}(\theta, \hat{r} = \hat{a})}{\hat{u}_0} = 2\sin\theta} \quad (36)$$

USE THIS IN (32) & (33) FOR
u_0 AND v_0