SINGULAR SOLNS. (GREEN'S FUNCTIONS) IN LOW REYNOLDS NUMBER FLOW PROBLEMS.

I A FEW KEY IDEAS:

- GREEN'S FN, $G(\underline{x},\underline{x}')$ OR SINGULAR SOLNS. PHYSICALLY REPRESENTS A FLOW'S RESPONSE, OBSERVED AT POINT $\underline{x}$, TO A POINT (VECTOR) FORCE APPLIED AT $\underline{x}'$. IT IS OFTEN CALLED A RESPONSE FN.

- SINGULAR SOLNS. CAN BE USED TO SOLVE LINEAR BOUNDARY VALUE PROBLEMS IN SIMPLE DOMAINS (i.e., BOUNDED BY COORDINATE SURFACES) OR IN COMPLEX DOMAINS. ANALYTICAL APPROACHES ARE POSSIBLE IN THE FIRST CASE WHILE NUMERICAL METHODS (e.g., BOUNDARY ELEMENT METHOD) MUST BE USED IN THE SECOND. BOUNDARY CONDITIONS ON THESE PROBLEMS CAN BE NONLINEAR.

- THE BASIC IDEA BEHIND THE USE OF SINGULAR SOLNS. (AT LEAST AS USED IN LOW-Re FLOW) IS TO BUILD SOLNS. FOR CASES WHEN FORCES ARE DISTRIBUTED IN SOME WAY WITHIN THE FLOW. THE EXAMPLES WE'LL LOOK AT (FLOW AROUND A SPHERE,

A LONGITUDINALLY TRANSLATING CYLINDER, AND A NORMALLY TRANSLATING CYLINDER) WILL SHOW HOW WE CAN FIRST IDEALIZE THE LOCAL DRAG FORCE ON THE BODY SURFACE AS A LOCAL POINT FORCE, AND SECOND, HOW WE CAN SUM UP THE FLOW SOLNS FOR EACH SUCH PT. FORCE (SUPERPOSITION) TO ARRIVE AT THE FINAL SOLN.

+ THESE METHODS WORK AND ARE USEFUL IN LOW-REYNOLDS NO. FLOWS BECAUSE THE GOVERNING EQNS. ARE LINEAR.

+ IN LOW REYNOLDS NO. FLOW, INERTIA (i.e., FLUID PARTICLE ACCELERATION) IS NEGLIGIBLE AND VISCOUS FORCES ARE DOMINANT. (SINCE PRESSURE FORCES ARE ALWAYS DETERMINED BY THE DOMINANT FORCE, IN THIS CASE FRICTION, PRESSURE FORCES ARE ALSO IMPORTANT.)

+ AS WILL BE SEEN, SINCE FLUID PTCLS. HAVE NEGLIGIBLE INERTIA, THE SUM OF FORCES ACTING ON EACH PARTICLE (GIVEN BY $-\nabla p$ AND $\mu \nabla^2 \underline{u}$) ARE AT EVERY INSTANT

IN <u>EQUILIBREUM</u> (i.e., SINCE $\vec{a}$ = acceleration $\doteq 0$, then BY NEWTON, $\Sigma \vec{F} = 0$).
SIMILARLY, SINCE FLUID PTCL'S <u>ROTATIONAL</u> <u>INERTIA</u> IS NEGLIGIBLE, THE <u>SUM OF</u> <u>MOMENTS</u> (GENERATED BY VISCOUS SHEAR STRESSES) ON ANY GIVEN FLUID PTCL. IS ALSO ZERO (AT ALL TIMES). [THESE TWO FACTS HOLD EVEN IN CASES WHERE, FOR EXAMPLE, THE DISTRIBUTED FORCES CHANGE CONTINUOUSLY IN TIME (eg., SUCH AS THOSE GENERATED BY SWIMMING MICROORGANISMS).]
THUS, LOW REYNOLDS NUMBER PROBLEMS ARE DYNAMICALLY EQUIVALENT TO <u>STATIC</u> FORCE PROBLEMS.

⊕ THE EXAMPLES TO BE CONSIDERED ALL ASSUME THAT THE TRANSLATING BODY MOVES IN AN INFINITE FLUID MEDIUM, SINCE THE EFFECT OF POINT FORCES ON THE VELOCITY FIELD DIES OFF AS $1/r$ (AS WILL BE SHOWN), AND SINCE THE MAGNITUDE OF THE POINT FORCES IS BY ASSUMPTION, SMALL

(LOCALIZED FORCE DISTRIBUTIONS OF LARGE MAGNITUDE WOULD INDUCE NON-NEGLIGIBLE FLUID ACCELERATION), THEN AN INFINITE FLOW MEDIUM ASSUMPTION IS TYPICALLY REASONABLE. CONSTRUCTION OF SINGULAR SOLNS. IN CASES WHERE BOUNDARY EFFECTS ARE IMPORTANT IS POSSIBLE, BUT USUALLY MORE DIFFICULT THAN IN INFINITE MEDIA.

(4) POINT FORCE SOLNS. CAN BE/ARE USED IN STUDYING FLOW IN PARTICLE-LADEN, LOW REYNOLDS NO. FLOWS, IN STUDYING PROPULSION OF SMALL ORGANISMS IN FLUID (e.g., SPERM, NEMATODES, CEPHALOPODS (SQUID), SMALL FISH, ORGANISMS W/ CILLIA).

WE'LL ILLUSTRATE HOW OUR 3 EXAMPLE CALCULATIONS CAN BE APPLIED TO UNDERSTANDING THE WHIP-LIKE UNDULATORY MOTION OF VARIOUS SMALL ORGANISMS.

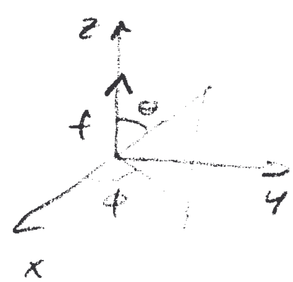
EXAMPLE 1 — FIND FLOW AND PRESSURE FIELDS PRODUCED BY A POINT FORCE (IN 3-D).
FIRST EXAMPLE OF SINGULAR SOLN. TO STOKES EQN.

MAIN POINT: THIS SOLN. WILL PROVIDE THE RESPONSE OF A 3-D LOW-Re NO. FLOW TO POINT FORCE IN AN INFINITE FLUID DOMAIN. WE CAN THEN USE THIS SINGULAR SOLN. TO BUILD FLOW SOLNS. DUE TO DISTRIBUTED FORCES IN AN INFINITE DOMAIN.

GOV. EQNS.

$$\nabla P = \mu \nabla^2 \underline{u} + f\,\underline{k}\,\delta(x)\,\delta(y)\,\delta(z) \qquad (1)$$

$$\nabla \cdot \underline{u} = 0 \qquad (2)$$



EQN. (1) IS STOKES EQN W/ A POINT FORCE OF MAGNITUDE f APPLIED IN THE + z DIRECTION AT $(x,y,z) = (0,0,0)$.

REMOVE P FROM (1):

$$\nabla \times (1) \Rightarrow \left| \; 0 = \mu \nabla^2 \underline{\omega} + \hat{\imath} f\, \delta(x)\, \delta'(y)\, \delta(z) - \hat{\jmath} f\, \delta'(x)\, \delta(y)\, \delta(z) \right. \qquad (3)$$

WHERE $\underline{\omega} = \nabla \times \underline{u}$ = VORTICITY

x-COMP. (3): $\mu \nabla^2 \omega_1 = -f\, \delta(x)\, \delta'(y)\, \delta(z)$ (4a)

y-COMP. (3): $\mu \nabla^2 \omega_2 = f\, \delta'(x)\, \delta(y)\, \delta(z)$ (4b)

LET $\boxed{\omega_1 = \dfrac{\partial \Omega}{\partial y} \qquad \omega_2 = -\dfrac{\partial \Omega}{\partial x}}$ (4c)

(4a) $\Rightarrow$ $\mu \nabla^2 \left(\dfrac{\partial \Omega}{\partial y} \right) = -f\, \delta(x)\, \delta'(y)\, \delta(z)$ (5a)

$\mu \nabla^2 \left(\dfrac{\partial \Omega}{\partial x} \right) = -f\, \delta'(x)\, \delta(y)\, \delta(z)$ (5b)

$\int (5a)\, dy \Rightarrow$ $\mu \nabla^2 \Omega = -f\, \delta(x)\, \delta(y)\, \delta(z) + h(x,z)$

$\int (5b)\, dx \Rightarrow$ $\mu \nabla^2 \Omega = -f\, \delta(x)\, \delta(y)\, \delta(z) + g(y,z)$ (6)

subtracting last 2-eqns $\Rightarrow$

$$h(x,z) = g(y,z)$$

THERE CAN BE NO x-dependence in h and no y-dependence in g SINCE THESE ARE THE SAME fn.

THUS, $h = g = \hat{f}(z)$. IT TURNS OUT THAT PHYSICALLY REALISTIC SOLNS. FOR $\underline{u}$ AND P FOLLOW BY SETTING $\boxed{\hat{f}(z) = 0}$. ALTERNATIVELY, (6) CAN BE READILY INTEGRATED BY CHOOSING

THUS,

$$\boxed{\mu \nabla^2 \Omega = -f\,\delta(x)\delta(y)\delta(z)} \quad (7)$$

NOW, SINCE THERE IS NO PREFERRED DIRECTION IMPLIED BY (7), i.e., NO θ OR ϕ DEPENDENCE IS SUGGESTED, THEN Ω SHOULD ONLY DEPEND ON r. THUS

$$(8) \quad \nabla^2 \Omega = \boxed{\frac{1}{r^2}\left(r^2 \Omega_{,r}\right)_{,r} = 0} \quad \text{for } r \neq 0$$

SOLVE FOR Ω USING DIVERGENCE THM.:

$$\int_{V} \nabla^2 \Omega \, dV = \int_{V} \nabla \cdot (\nabla \Omega) \, dV = \oint_{S} \nabla \Omega \cdot \underline{n} \, ds$$

$$= -\int_{V} f\,\delta(x)\delta(y)\delta(z)\, dV = -f/\mu$$

WHERE V IS A SPHERE OF RADIUS r and S IS THE SURFACE OF THE SPHERE. SINCE $\underline{n} = \hat{e}_r$,

$\nabla \Omega = \frac{\partial}{\partial r} \Omega \hat{e}_r$, AND $ds = r^2 \sin\theta \, d\phi \, d\theta$

THEN,

$$\oint_S \nabla\Omega\cdot\underline{n}\,ds = \int_0^{2\pi}\int_0^{\pi}\left(\frac{\partial\Omega}{\partial r}\right) r^2\sin\theta\,d\theta\,d\phi = -f/\mu$$

$$= \left(\frac{\partial\Omega}{\partial r}\right) r^2 \cancel{(-\cos\theta)}^{(2)}\Big|_0^{\pi}(2\pi)$$

$$= 4\pi r^2\,\Omega_{,r} = -\frac{f}{\mu}$$

$$\Rightarrow \quad \Omega_{,r} = -\frac{f}{\mu 4\pi r^2}$$

$$\Rightarrow \quad \Omega = \frac{f}{4\pi r\mu} + C_0$$

SINCE WE ARE ONLY INTERESTED IN DERIVATIVES OF Ω, WE CAN SET $C_0 = 0$ W/OUT LOSS OF GENERALITY,

$$\Rightarrow \quad \boxed{\Omega = \frac{f}{4\pi r\mu}} \qquad (9)$$

NOW WE CAN FIND $\underline{u}$ AND p:
FIRST NOTE THAT

$$\underline{\omega} = \omega_1\hat{i} + \omega_2\hat{j} = \frac{\partial\Omega}{\partial y}\hat{i} - \frac{\partial\Omega}{\partial x}\hat{j}$$

$$= \nabla\times(\Omega\underline{k})$$

THUS, $\quad \nabla\times\underline{u} = \nabla\times(\Omega\underline{k}) \qquad (10)$

NOW IT IS TEMPTING TO SAY THAT

$\underline{u} = \Omega \underline{k}$, HOWEVER, IT IS FOUND THAT $\nabla \cdot \Omega \underline{k} \neq 0$ AS RQD. BY CONT'Y. THUS.

$$\underline{u} = \Omega \underline{k} + \underline{F}(\underline{x}) \qquad (11)$$

BUT, SINCE

$$\underline{\omega} = \nabla \times \underline{u} = \nabla \times (\Omega \underline{k})$$

THEN

$$\nabla \times \underline{F} = 0$$

$$\Rightarrow \boxed{\underline{F} = \nabla \phi}$$

$$\Rightarrow \quad \underline{u} = \Omega \underline{k} + \nabla \phi \qquad (12)$$

NOW FROM CONT'Y

$$\nabla \cdot \underline{u} = \nabla \cdot (\Omega \underline{k}) + \nabla^2 \phi = 0$$

$$\Rightarrow \nabla^2 \phi = -\frac{\partial \Omega}{\partial z} = \frac{f z}{4 \pi \mu r^3}$$

(NOTE $r = (x^2 + y^2 + z^2)^{1/2}$

SINCE THIS LAST EQN IS DIFFICULT TO SOLVE, LET'S GO BACK AND DEFINE

$$\boxed{\underline{u} = \Omega \underline{k} + \nabla \left(\frac{\partial \hat{\Phi}}{\partial z} \right)} \qquad (12a)$$

$$\Rightarrow \quad \nabla \cdot \underline{u} = \frac{\partial \Omega}{\partial z} + \nabla^2 \hat{\Phi}_{/z} = \boxed{\frac{\partial}{\partial z}\left(\Omega + \nabla^2 \hat{\Phi}\right) = 0}$$

$$\Rightarrow \quad \Omega + \nabla^2 \hat{\Phi} = 0 \qquad (13)$$

NOTE WE CHOOSE THE FN. THAT ARISES DURING INTEGRATION TO BE 0 SINCE IT LEADS TO A PDE IN $\hat{\Phi}$ WHICH WE CAN EASILY SOLVE. WE HAVE THIS FREEDOM SINCE WE ARE ONLY SEEKING A SOLN. OF (1) AND (2).

THE ONLY CONDITION THIS SOLN. MUST SATISFY IS THAT $\underline{u} \to 0$ AS $r \to \infty$.

$$(13) \Rightarrow \quad \nabla^2 \hat{\Phi} = -\Omega = \frac{-f}{4\pi\mu r} \qquad (14)$$

BY INSPECTION, A SOLN. FOR (14) (WHICH GOES TO 0 AS $r \to \infty$) IS OBTAINED BY ASSUMING $\hat{\Phi} = f_n(r)$

$\Rightarrow$

$$\frac{1}{r^2}\left(r^2 \hat{\Phi}_{,r}\right)_{,r} = -\frac{f}{4\pi\mu r}$$

$$\Rightarrow r^2 \Phi_{,r} = -\frac{f r^2}{8\pi\mu} \leftarrow$$

(TAKE CONSTS. AS 0 SINCE WE ONLY WANT DERIVS OF Φ)

$$\Rightarrow \Phi_{,r} = -\frac{f}{8\pi\mu}$$

$$\Rightarrow \boxed{\Phi = -\frac{f r}{8\pi\mu}} \qquad (15)$$

$$\Rightarrow \quad \nabla \hat{\Phi}_{1z} = \nabla\left(\frac{-fz}{8\pi\mu r}\right)$$

$$= \frac{f}{8\pi\mu}\left[\frac{xz}{r^3},\ \frac{yz}{r^3},\ \frac{-1}{r} + \frac{z^2}{r^3}\right]$$

$$= \frac{f}{8\pi\mu}\left[\frac{xz}{r^3},\ \frac{yz}{r^3},\ \frac{-(x^2+y^2)}{r^3}\right]$$

$$\Rightarrow \quad \underline{u} = \Omega \underline{k} + \nabla\hat{\Phi}_{1z}$$

$$\boxed{\underline{u} = \frac{f}{8\pi\mu}\left[\frac{xz}{r^3},\ \frac{yz}{r^3},\ \frac{1}{r} + \frac{z^2}{r^3}\right]} \qquad (15)$$

OR MORE GENERALLY,

$$u_i = \frac{f}{8\pi\mu}\left[\frac{\delta_{ij}}{r} + \frac{x_i x_j}{r^3}\right] \qquad (16)$$

FOR FORCE ACTING IN j^{th} direction ($j = 1, 2, 3$) (*)

STOKESLET

SEE ATTACHMENT (FROM LIGHTHILL, MATHEMATICAL BIOFLUIDDYNAMICS) FOR ALTERNATIVE DERIVATION OF (15) (OR EQUIVALENTLY (16)).

THE FLOW SOLN. IN (16) IS CALLED A STOKESLET, IT'S AN IMPORTANT BUILDING BLOCK FOR CONSTRUCTING CREEP FLOWS SUBJECT TO DISTRIBUTED LOADING.

PRESSURE FIELD:

TAKE $\nabla\cdot$ OF (1) TO GET

$$\nabla^2 p = \mu \nabla^2 \overset{0}{(\nabla\cdot\underline{u})} + f\,\delta(x)\,\delta(y)\,\delta'(z)$$

$$\Rightarrow \boxed{\nabla^2 P = f\,\delta(x)\,\delta(y)\,\delta'(z)} \qquad (17)$$

USE SAME APPROACH AS BEFORE AND LET

$$\boxed{P = \underline{P}_{,z}}$$

$(17) \Rightarrow$

$$\nabla^2 \frac{\partial \underline{P}}{\partial z} = f\,\delta(x)\,\delta(y)\,\delta'(z)$$

INTEGRATE $\Rightarrow$ $$\boxed{\nabla^2 \underline{P} = f\,\delta(x)\,\delta(y)\,\delta(z)} \qquad (18)$$

(AGAIN USE FREEDOM OF NOT HAVING B.C.'S ON $\underline{P}$, OR EQUIVALENTLY P, TO SET FN. FROM INTEGRATION TO ZERO.)

AGAIN, SINCE NO PREFERRED DIRECTION IS INDICATED BY (18), $\underline{P} \neq fn(\theta)$ and $\underline{P} \neq fn(\phi)$. $\underline{P}$ only depends on r:

$$\nabla^2 \underline{P} = \frac{1}{r^2}\left(r^2 \underline{P}_{,r}\right)_{,r} = f\,\delta(x)\,\delta(y)\,\delta(z)$$

$$\Rightarrow \int_V \nabla^2 \underline{P}\, dV = \int_V \nabla\cdot\nabla\underline{P}\, dV = \oint_S \nabla\underline{P}\cdot\underline{n}\, dS = \int_V f\,\delta(x)\,\delta(y)\,\delta(z)\, dV$$

V = sphere of radius r
S = surface of sphere
$\underline{n} = \underline{e}_r$
$dS = r^2 \sin\theta\, d\theta\, d\phi$

REPEATING INTEGRATIONS DONE ON pg. 8

WE OBTAIN

$$\underline{p}_{,r}\, 4\pi r^2 = f$$

$$\Rightarrow \quad \underline{p} = -\frac{f}{4\pi r}$$

(SET INTEGRATION CONSTANT = 0; ONLY INTERESTED IN GRADIENTS IN $\underline{p}$)

$$\Rightarrow \quad \boxed{p = \underline{p}_{,z} = \frac{fz}{4\pi r^3}} \qquad (19)$$

$$r = (x^2+y^2+z^2)^{1/2}$$

PRESSURE FIELD FOR STOKESLET.

II SOLN. OF LOW REYNOLDS NO. EXTERNAL FLOWS (ABOUT SOLID OBJECTS) USING SUPERPOSITION

IN THE PRECEEDING, WE OBTAINED A SINGULAR SOLN. FOR FLOW INDUCED W/IN AN INFINITE MEDIUM DUE TO A POINT FORCE AT THE ORIGIN. WE CAN NOW BUILD SOLNS. FOR FLOWS PRODUCED BY A DISTRIBUTION OF POINT FORCES ON A SOLID BODY.

AS MENTIONED, THIS APPROACH RESTS ON THE ASSUMPTION (IDEALIZATION) THAT A LOCAL SHEAR FORCE ($\equiv \tau\, dA$ where τ is local shear stress and dA is an element of area)

ON THE BODY CAN BE MODELED AS A POINT FORCE. THIS IS A REASONABLE ASSUMPTION SINCE WE TAKE dA TO BE INFINITESIMALLY SMALL. AS WILL BE SEEN IN THE FOLLOWING EXAMPLES, NO-SLIP AND NO-PENETRATION B.C.'S ON SOLID BOUNDARIES GENERALLY ARE NOT BOTH SATISFIED W/ THE STOKESLET SOLN. ALONE. IN ORDER TO SATISFY THE B.C.'S, A PARTICULARLY USEFUL <u>POTENTIAL FLOW</u> SOLN DUE TO A <u>DIPOLE</u> CAN OFTEN BE USED. [RECALL THAT <u>POTENTIAL FLOW</u> IS FLOW IN WHICH VISCOUS FORCES <u>AND</u> VORTICITY ARE NEGLIGIBLE (IN INCOMPRESSIBLE FLOW, INVISCID FLOW IS EQUIVALENT TO NON-VORICAL FLOW SINCE $\nabla^2 \underline{u} = \nabla(\cancelto{0}{\nabla\cdot\underline{u}}) - \nabla\times(\nabla\times\underline{u})$). THUS, SINCE $\nabla\times\underline{u} = 0$ THEN $\underline{u}$ CAN BE CALCULATED AS THE GRADIENT OF A SCALAR FN., THE VELOCITY POTENTIAL ϕ: $\underline{u} = \nabla\phi$.] <u>ANY</u> <u>POTENTIAL FLOW</u> SOLN, $\underline{u}_p$, CAN BE ADDED TO <u>ANY</u> SOLN. OF STOKES EQN. SINCE $\nabla\times(\nabla P) = 0 = \mu\nabla^2(\nabla\times\underline{u}) + \nabla\times\underline{F}$ (WHERE $\underline{F}$ REPRESENTS E.G. A DISTRIBUTED FORCE -- THIS TERM IS <u>USUALLY</u> NOT PRESENT).

IN PARTICULAR, WE HAVE

$$0 = \mu \nabla^2 (\nabla \times (\underline{u}_s + \underline{u}_p)) + \nabla \times \underline{F}$$

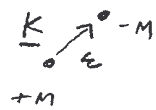
WHERE $\underline{u}_s$ IS THE SOLN. TO

$$0 = \mu \nabla^2 (\nabla \times \underline{u}_s) + \nabla \times \underline{F}$$

(i.e., $\underline{u}_s$ IS THE VISCOUS FLOW SOLN.) AND $\underline{u}_p = \nabla \phi$. SINCE $\nabla \times \underline{u}_p = \nabla \times (\nabla \phi) = 0$, THEN IT IS CLEAR THAT <u>ANY</u> <u>POTENTIAL FLOW</u> SOLN. CAN BE ADDED TO $\underline{u}_s$. THE <u>DIPOLE</u> IS ONE OF <u>MANY</u> POTENTIAL FLOW SOLNS, (SEE e.g., MILNE-THOMPSON, THEORETICAL HYDRODYNAMICS) AND IS GIVEN BY

$$(20a) \qquad \boxed{\underline{u}_p = \underline{K} \cdot \nabla \left(\frac{r}{4\pi} \right)} \qquad r = (x^2+y^2+z^2)^{1/2}$$

WHERE $\underline{K}$ IS THE (VECTOR) STRENGTH OF THE DIPOLE. [A DIPOLE IS DERIVED FROM A <u>POTENTIAL FLOW</u> <u>SOURCE</u> AND <u>SINK</u>, HAVING EQUAL STRENGTH AND SEPARATED BY AN INFINITESIMAL DISTANCE IN THE DIRECTION OF $\underline{K}$.



M = source strength
ε = sepn distance
See Panton, Incompressible Flow for physical derivation.

A QUICKER DERIVATION IS TO START w/ THE POTENTIAL FOR A POINT VOLUME SOURCE

$$\phi = -\frac{1}{r} \qquad (20)$$

LET $\underline{K}$ BE DIRECTED IN THE DIRECTION OF x_i ($i=1, 2,$ or 3) AND DIFFERENTIATE (20) WRT x_i:

$$\boxed{\underset{\sim}{\phi} = \frac{\partial \phi}{\partial x_i} = \frac{x_i}{r^3}} \qquad (21)$$

WHERE $\underset{\sim}{\phi}$ IS THE vel. POTENTIAL FOR A DIPOLE DIRECTED IN THE DIRECTION OF x_i. THUS, SINCE $\underline{u}_{DIPOLE} = \nabla \underset{\sim}{\phi}$

$$\boxed{u_{j_{DIPOLE}} = \frac{\partial \underset{\sim}{\phi}}{\partial x_j} = \frac{\delta_{ij}}{r^3} - \frac{3 x_i x_j}{r^5}} \qquad (22)$$

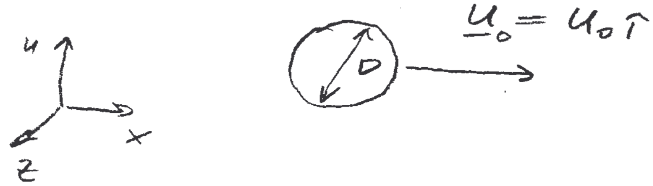
$\boxed{j = 1, 2, 3}$ | DIPOLE ALIGNED IN i^{th} DIRECTION |

FOR A DIPOLE OF STRENGTH K (22) BECOMES

$$\boxed{u_{j_{DIPOLE}} = K\left[\frac{\delta_{ij}}{r^3} - \frac{3 x_i x_j}{r^5}\right]} \qquad (22a)$$

| j = 1, 2, 3 DIPOLE IN ...

EXAMPLE - USE SUPERPOSITION TO DETERMINE THE FLOW FIELD AND DRAG DUE TO A SMALL SPHERE TRANSLATING IN A FIXED DIRECTION (SAY x).



BEFORE ANALYZING THIS PROBLEM, LET'S BE CLEAR ON WHEN A STOKE'S FLOW MODEL IS APPROPRIATE. STOKES FLOW (i.e., LOW REYNOLDS NUMBER FLOW) EXISTS WHEN THE FLOW'S CHARACTERISTIC REYNOLDS NUMBER

$$Re = \frac{u_0 L}{\nu} << 1$$

IN ORDER TO IDENTIFY THE APPROPRIATE LENGTH L, WE HAVE TO RECOGNIZE THAT

$$Re \approx \frac{\text{DOMINANT INERTIAL TERM}}{\text{DOMINANT VISCOUS TERM}} .$$

IN THE PRESENT EXAMPLE, WE INTUITIVELY EXPECT (GUESS) THAT THE DOMINANT INERTIAL TERM IS FROM THE x-MOMENTUM

EQN:
$$\rho u u_{,x} \approx \rho \frac{U_0^2}{D}$$

WHERE WE CHOOSE D AS THE X-LENGTH SCALE SINCE D IS THE ONLY LENGTH IN THE PROBLEM (MORE SPECIFICALLY, WE EXPECT THAT SIGNIFICANT CHANGES IN u OCCUR OVER LENGTHS ON THE ORDER OF D).

THE DOMINANT VISCOUS TERM (WHICH MUST COME FROM THE X-MOM. EQN. SINCE WE DON'T WANT WANT TO COMPARE INERTIA IN ONE DIRECTION W/ A FRICTION FORCE IN ANOTHER DIRECTION) IS
$$\mu u_{,yy} \approx \mu \frac{U_0}{D^2}. \quad \text{THUS,}$$

$$Re \approx \frac{\rho u u_{,x}}{\mu u_{,yy}} \approx \frac{\rho U_0 D}{\mu}$$

THUS, IN ORDER FOR THE STOKES FLOW MODEL TO BE APPROPRIATE IN THE PRESENT EXAMPLE

$$\boxed{\frac{\rho U_0 D}{\mu} << 1} \qquad (23)$$

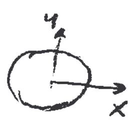
NOTICE THAT BASED ON THE CRITERION IN (23), WE CAN USE THE FOLLOWING ANALYSIS EVEN IN CASES WHERE D IS RELATIVELY LARGE — WE JUST HAVE TO BE SURE THAT EITHER U_0 IS SMALL ENOUGH AND/OR $\nu = \mu/\rho$ IS BIG ENOUGH THAT (23) HOLDS.

NOW TO THE EXAMPLE.

REFERING TO THE STOKESLET SOLN. IN (16)

$$u_{i_s} = \frac{f}{8\pi\mu}\left[\frac{\delta_{ij}}{r} + \frac{x_i x_j}{r^3}\right] \qquad (16)$$

WE HAVE TWO CONDITIONS THAT THE SOLN. $\underline{u}$ MUST SATISFY:



$$\boxed{\underline{u} = U_0 \hat{\imath}} \qquad (24) \qquad \boxed{\text{on } r = a = \frac{D}{2}}$$

(NO SLIP CONDITION)

AND

NO PENETRATION CONDITION $\boxed{\underline{u}\cdot\underline{n} = \underline{u}\cdot\underline{e}_r = U_0\hat{\imath}\cdot\underline{e}_r}$ (25) $\boxed{r=a}$

NOTE — WE ARE DEFINING THE FRAME OF REFERENCE SO THAT THE CENTER OF THE SPHERE IS PASSING OVER THE ORIGIN AT THE INSTANT THAT WE IMPOSE (24) AND (25). THIS APPROACH IS POSSIBLE SINCE THE FLOW IS DYNAMICALLY STATIC (i.e. no acceleration anywhere)

AT ALL INSTANCES, IF THIS WEREN'T THE CASE WE'D HAVE TO ATTACH REF. FRAME TO SPHERE (IN ORDER TO AVOID TIME DEPENDENT SOLN).

FROM (25), WE SEE THAT

$$(\underline{u} - U_0\hat{\imath})\cdot\hat{e}_r = 0 \quad \text{AT } r=a$$

$$\Rightarrow \quad \underline{u} = U_0\hat{\imath} \qquad r=a$$

THUS, THE NO-PENETRATION COND. (25) IS REDUNDANT W/ RESPECT TO THE NO-SLIP CONDITION (24). THIS IS DUE TO THE WAY WE'VE DEFINED OUR COORD. SYS. AND DUE TO THE STATIC NATURE OF THE FLUID DYNAMICS.

NOW, ASSUME POINT FORCE ACTS IN X-DIR,

$$(16)\Rightarrow \quad \underline{u}_s(r=a) = \frac{f}{8\pi\mu}\left[\frac{\underline{e}_1}{a} + \frac{x_1}{a^3}(x_i\underline{e}_i)\Big|_{r=a}\right] \qquad (26)$$

SINCE THIS DOESN'T SATISFY (24) (DUE TO POSITION DEPENDENCE x_1), WE NEED TO SUPERPOSE A SECOND SOLN. TO GET RID OF SECOND TERM IN (26).

NOW, THE VELOCITY FIELD FOR A POTENTIAL DIPOLE ALIGNED IN THE X-DIR IS FROM (22)

$$\underline{u}_D\Big|_{r=a} = K\left(\frac{\underline{e}_1}{a^3} - \frac{3x_1(x_i\underline{e}_i)}{a^5}\Big|_{r=a}\right) \qquad (27)$$

WHERE K IS THE MAGNITUDE OF $\underline{K}$ IN (22a) ON pg. 16.

COMPARING (26) AND (27), WE SEE THAT WE CAN CANCEL THE POSITION-DEP. TERM IF WE ADD $\underline{u}_s$ AND $\underline{u}_0$ AND CHOOSE K SO THAT

$$\frac{f}{8\pi\mu a^3}\, x_1 (x_i \underline{e}_i)\Big|_{r=a} = \frac{K(3)}{a^5}\, x_1 (x_i \underline{e}_i)\Big|_{r=a}$$

I.E., LET

$$\boxed{K = \frac{f a^2}{24\pi\mu}} \qquad (28)$$

THUS,

$$(\underline{u}_s + \underline{u}_0)\big|_{r=a} = \left[\frac{f}{8\pi\mu a} + \frac{f a^2}{24\pi\mu a^3}\right]\underline{e}_1 = u_0\, \underline{e}_1$$

HENCE

$$\boxed{f = 6\pi\mu a u_0} \qquad (29)$$

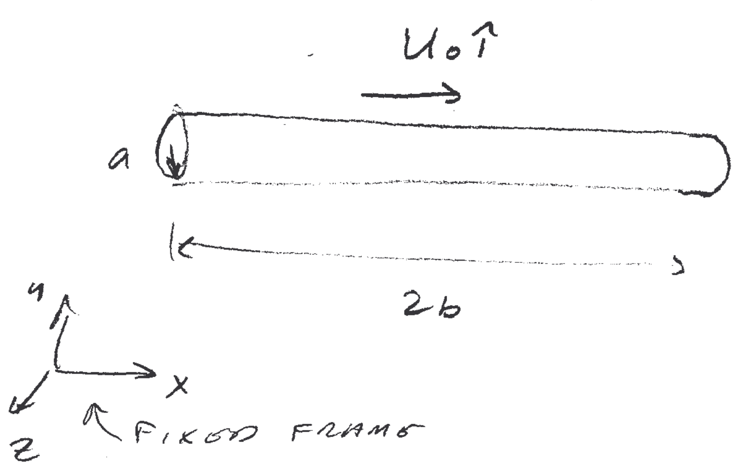
AND THE FINAL SOLN. IS

$$\boxed{\begin{aligned} \underline{u} = \underline{u}_s + \underline{u}_0 &= \frac{f}{8\pi\mu}\left[\frac{\underline{e}_1}{r} + \frac{x_1 r \underline{e}_r}{r^3}\right] \\ &\quad + \frac{f a^2}{24\pi\mu}\left[\frac{\underline{e}_1}{r^3} - \frac{3 x_1 r \underline{e}_r}{r^5}\right] \\ &\text{WHERE } f = 6\pi\mu a u_0 \end{aligned}} \qquad (30)$$

NOTE, WE USED $x_i \underline{e}_i = r \underline{e}_r$ ABOVE

NOTE THAT (29) IS <u>STOKE'S</u> <u>DRAG LAW</u>.
AND THAT f IS THE TOTAL DRAG
ON THE SPHERE. THIS RESULT
AND CALCULATION OF THE FLOW FIELD
CAN ALSO BE CARRIED OUT USING STANDARD
SEPARATION OF VARIABLES.

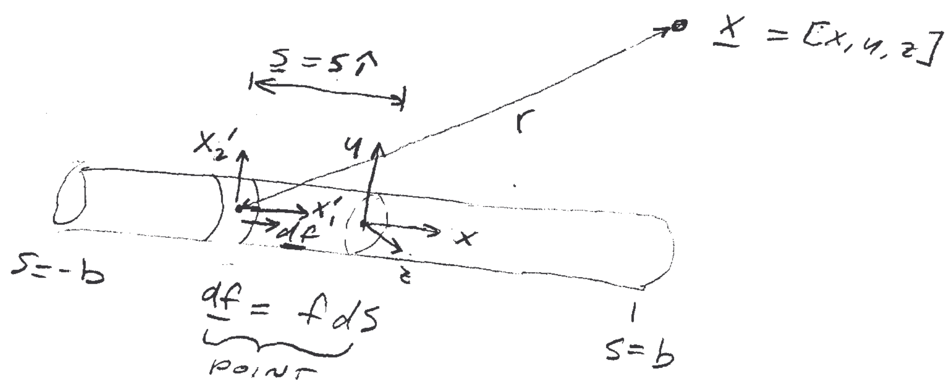
<u>EXAMPLE</u> – DETERMINE THE APPROXIMATE
DRAG ON A SLENDER ROD
IN THE CASE WHERE THE ROD
MOVES TANGENTIALLY IN THE DIRECTION
OF IT'S LONG AXIS. IN PARTICULAR,
DETERMINE D (= Drag) when $b/a \gg 1$
WHERE $2b$ = rod length and
a = rod radius.



KEY POINT: THIS SOLN. AND THE ONE IN THE NEXT EXAMPLE CAN BE USED TO EXPLAIN / INTERPRETE UNDULATORY PROPULSION (AS EXPLAINED IN ATTACHMENT).

⊕ SEE HAPEL AND BRENNER (pp 45, 156) FOR EXACT SOLNS TO FLOW PRODUCED BY TANGENTIALLY AND NORMALLY TRANSLATING, FINITE LENGTH RODS.

SOLN: MODEL LOCAL ACTION OF TANGENTIAL SHEAR STRESS FROM ROD ON FLUID AS A TANGENTIALLY ACTING POINT FORCE



$\underbrace{d\underline{f}}_{\text{POINT FORCE ACTING AT POSITION } s \text{ on rod}} = f\,ds$

FROM (15) OR (16) (pg 11) WE CAN WRITE DOWN THE velocity produced AT POSITION $\underline{x}$ BY THE POINT FORCE ACTING AT $\underline{x} = s\hat{\imath}$ AS

$$\underline{u}(\underline{x}) = \frac{df}{8\pi\mu}\left[\frac{1}{r}\hat{\imath} + \frac{x_1'}{r^3}\underbrace{(x_1'\hat{e}_1 + x_2'\hat{e}_2 + x_3'\hat{e}_3)}_{\underline{x}'}\right]$$

$df = f\,ds$ (NOTE $f \doteq$ Force/Length)

$x_1' = x - s$

$x_2' = y \qquad \Rightarrow \underline{x}' = \underline{x} - \underline{s}$

$x_3' = z \qquad = [x-s, y, z] \leftarrow$ [ ] Denotes a vector

$r = [(x_1')^2 + (x_2')^2 + (x_3')^2]^{1/2} = |\underline{x}'|$

$= [(\underline{x}-\underline{s})\cdot(\underline{x}-\underline{s})]^{1/2} = |\underline{x}-\underline{s}|$

$\underline{s} = s\hat{\imath} \qquad \underline{x} = x\hat{e}_1 + y\hat{e}_2 + z\hat{e}_3$

$$\Rightarrow \boxed{\underline{u}(\underline{x}) = \frac{f\,ds}{8\pi\mu}\left[\frac{\hat{\imath}}{|\underline{x}-\underline{s}|} + \frac{(x-s)(\underline{x}-\underline{s})}{|\underline{x}-\underline{s}|^3}\right]} \quad (31)$$

THUS, IF WE WANT TO DETERMINE THE VELOCITY PRODUCED BY SUCH POINT FORCES WE SIMPLY INTEGRATE (31);

$$\boxed{\underline{u}(s) = \frac{f}{8\pi\mu}\int_{-b}^{b}\left[\frac{\hat{\imath}}{|\underline{x}-s\hat{\imath}|} + \frac{(x-s)(\underline{x}-\underline{s})}{|\underline{x}-s\hat{\imath}|^3}\right]ds} \quad (32)$$

NOTE THAT (32) REPRESENTS 3 INTEGRALS. ALSO NOTE THAT $\underline{x}$ (and THUS x, y, AND z) ARE PARAMETERS (CONSTANTS) IN THE INTEGRALS. LET'S USE (32) TO GET FLOW FIELD -- THEN WE'LL GET DRAG.

<u>X-DIR:</u>

$$u_1 = \frac{f}{8\pi\mu}\int_{-b}^{b} \frac{1}{\sqrt{r_0^2 - 2xs + s^2}}\, ds \; +$$

$$\frac{f}{8\pi\mu}\int_{-b}^{b} \frac{(x-s)^2}{(r_0^2 - 2xs + s^2)^{3/2}}\, ds$$

USE INTEGRAL TBLS

$r_0^2 = x^2 + y^2 + z^2$

$$\boxed{\begin{aligned} \Rightarrow u_1 = \frac{f}{8\pi\mu}\Bigg[\!\!\Bigg[& \overset{\text{I}}{2\ln\left|2s - 2x + 2(s^2 - 2xs + r_0^2)^{1/2}\right|}\Bigg|_{-b}^{b} \\ & + \overset{\text{II}}{\frac{(4s - 4x)(x^2 - r_0^2)}{(4r_0^2 - 4x^2)(s^2 - 2xs + r_0^2)^{1/2}}}\Bigg|_{-b}^{b} \\ & + \frac{(-4xs + 4r_0^2)\underbrace{(\cancel{-2x} + 2x)}_{0}}{(4r_0^2 - 4x^2)(s^2 - 2xs + r_0^2)^{1/2}}\Bigg|_{-b}^{b}\ \Bigg]\!\!\Bigg] \end{aligned}} \qquad (33)$$

$\boxed{r_0^2 = x^2 + y^2 + z^2}$ | x-COMP OF vel. field

SINCE u_1 CLEARLY VARIES w/ X ON ROD SURFACE, WE'LL SEEK AN APPROXIMATE ANSWER FOR DRAG BY DETERMING f (WHICH IS GIVEN BY $2bf = D$) NEAR WERE ASSUMING CENTER ($\frac{|x|}{b} \ll 1$):

$$\textcircled{I} = 2\ln \frac{\left|2b - 2x + 2b\left(1 - 2\frac{x}{b} + \left(\frac{r_0}{b}\right)^2\right)^{1/2}\right|}{\left|-2b - 2x + 2b\left(1 + 2\frac{x}{b} + \left(\frac{r_0}{b}\right)^2\right)^{1/2}\right|}$$

$$\boxed{\text{For: } \frac{|x|}{b} \ll 1 \quad \text{AND} \quad \left(\frac{r_0}{b}\right)^2 \ll 1}$$

TAYLOR EXPAND FOR $\pm 2\left(\frac{x}{b}\right) + \left(\frac{r_0}{b}\right)^2 \ll 1$

$$\sqrt{1 - 2\left(\frac{x}{b}\right) + \left(\frac{r_0}{b}\right)^2} = 1 + \frac{1}{2}\left(-2\left(\frac{x}{b}\right) + \left(\frac{r_0}{b}\right)^2\right) + \text{H.O.T.}$$

$$\sqrt{1 + 2\left(\frac{x}{b}\right) + \left(\frac{r_0}{b}\right)^2} = 1 + \frac{1}{2}\left(2\left(\frac{x}{b}\right) + \left(\frac{r_0}{b}\right)^2\right) + \text{H.O. Terms}$$

$$\Rightarrow \textcircled{I} = 2\,\frac{\ln\left|2b\left(1 - \left(\frac{x}{b}\right) + 1 + \frac{1}{2}\left(-2\left(\frac{x}{b}\right)\right) + \frac{1}{2}\left(\frac{r_0}{b}\right)^2\right)\right|}{\ln\left|-2b + 2x + 2b + 2b\left(\frac{1}{2}\right)\left(\frac{2x}{b} + \left(\frac{r_0}{b}\right)^2\right)\right|}$$

$$\approx 2\ln\left|(4b)\right| \Big/ \left|\left(4x + \frac{r_0^2}{b}\right)\right|$$

$$\boxed{\textcircled{I} = 2\ln \frac{4b^2}{[4xb + x^2 + y^2 + z^2]}} \qquad (34)$$

IF WE NOW SET $x = 0$ (or ASSUME $\frac{x}{b} \ll 1$) AND LET $y^2 + z^2 = a^2$, i.e., EVALUATE $\textcircled{I}$ AT THE center SURFACE OF THE ROD, WE OBTAIN THE SECOND TERM IN u_1 GIVEN IN EQN. (30) OF HANDOUT.

$$\text{I} \ @\ x=0,\ y^2+z^2=a^2 \ = \ \boxed{2 \ln \frac{4b^2}{a^2}} = \boxed{\text{I}} \quad (34)$$

NOW LOOK AT II IN (33)

$$\text{II} = \left. \frac{(x-s)}{(s^2-2xs+r_0^2)^{1/2}} \right|_{-b}^{b}$$

SET $x=0$ AND $r_0^2 = \cancel{x^2}^{\,0} + y^2+z^2 = a^2$

$$\Rightarrow \ \text{II} = \frac{-2b}{\sqrt{b^2+a^2}} = \frac{-2b}{b(1+(\frac{a}{b})^2)^{1/2}} = \frac{-2}{1+\frac{1}{2}(\frac{a}{b})^2 + \text{H.O.T.}}$$

$$\boxed{\text{II} \approx -2 + O\left(\left(\frac{a}{b}\right)^2\right)} \quad (35)$$

THUS, USING (35) & (34) IN (33)

WE GET

$$\boxed{u_1 \approx \frac{f}{8\pi\mu}\left(2 \ln\left(\frac{4b^2}{a^2}\right) - 2\right)} \quad (36)$$

VALID FOR $\frac{|x|}{b} << 1$ &

$\left(\frac{a}{b}\right) << 1$ AND EVALUATED

ON SURFACE OF ROD. $(y^2+z^2=a^2)$

RATHER THAN DOING THE REMAINING TWO INTEGRALS IN (32) (FOR u_2 & u_3),

SINCE WE ARE ONLY INTERESTED IN u_2 & u_3 @ THE ROD SURFACE WE RECOGNIZE THAT THESE ARE BOTH ZERO ON SURFACE (SINCE $\underline{u}$ @ surface $= U_0 \hat{i}$), AND PRESUME THE INTEGRALS WOULD SHOW THIS.

NOW TO DETERMINE THE APPROXIMATE DRAG, WE USE THE KNOWN B.C. ON u_1 : $u_1(\text{SURFACE}) = U_0$

THUS, USING (36) AND SOLVING FOR f WE GET

$$\boxed{f = \frac{U_0 2\pi\mu}{\left(\ln\left(\frac{b}{a}\right) + 1\right)}} \qquad (37)$$

SINCE $D \approx 2bf$, THEN

$$\boxed{D \approx \frac{4\pi\mu b U_0}{\ln\left(\frac{b}{a}\right) + 1}} \qquad (38)$$