LINEAR STABILITY CONTINUED –

CAPILLARY INSTABILITY OF A CYLINDRICAL JET

THIS ANALYSIS WILL EXPLAIN AND QUANTIFY THE BREAK-UP OF A LIQUID JET, e.g., SEEN WHEN A THIN STREAM OF WATER FLOWS LAMINARLY FROM A TAP.

STRATEGY / KEY POINTS

1) CALCULATE OR GUESS BASIC STATE – (MUST SATISFY MOMENTUM & CONTINUITY AND APPROPRIATE BOUNDARY CONDITIONS ON BASIC FLOW)

2) STATE GOVERNING EQNS (MOMENTUM & CONTY) AND BOUNDARY CONDITIONS

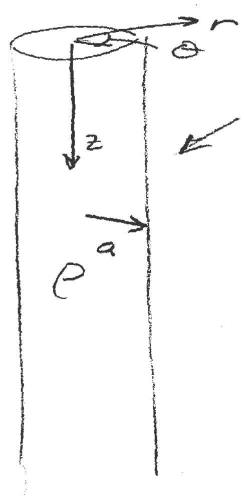
3) ASSUME PERTURBATIONS TO THE BASIC FLOW ARE OF NORMAL MODE VARIETY

4) LINEARIZE GOVERNING EQNS AND BOUNDARY CONDITIONS. THIS IS PERMISSIBLE SINCE WE ARE FOCUSING (LIMITING) ATTENTION ON INITIAL, LOW AMPLITUDE PERTURBATIONS TO THE BASIC FLOW.

5) ATTEMPT TO SOLVE LINEARIZED SYSTEM OF EQNS. IN ORDER TO DETERMINE THE GROWTH RATE, $s = \sigma + i\omega$. THIS RELATIONSHIP (AN EIGENVALUE RELATIONSHIP) WILL DEPEND ON VARIOUS PROBLEM PARAMETERS (AS IN KELVIN-HELMHOLTZ); DETERMINE STABILITY OF BASIC FLOW TO NORMAL MODE DISTURBANCES BASED ON THE EIGENVALUE RELATIONSHIP DETERMINED.

BASIC FLOW

ASSUME (GUESS) THAT FLOW FIELD IN UNPERTURBED JET IS GIVEN BY



$\underline{u} = U_o \hat{e}_z$

$a =$ undisturbed jet radius

$\underline{u} = U_o \hat{e}_z$ WHERE $U_o =$ CONSTANT. ASSUME VISCOUS FORCES ARE NEGLIGIBLE WITHIN JET. IN REALITY, IF WE LOOK AT THE CASE WHERE

SAY WATER EXITS FROM A TAP, IT IS CLEAR THAT VORTICITY ($\nabla \times \underline{u}$) IS CREATED AT THE PERIPHERY OF THE LIQUID AS IT LEAVES THE TAP. AS DESCRIBED BY THE VORTICITY EQN., $\rho \frac{D\underline{w}}{Dt} = \mu \nabla^2 \underline{w}$, THIS VORTICITY (LARGELY IN θ-DIRECTION) WILL DIFFUSE INWARD TOWARD THE CENTER OF THE JET. WE CAN ESTIMATE HOW FAST THIS INWARDLY DIFFUSING VORTICITY WILL DIFFUSE TO THE CENTER BY ARGUING THAT RADIAL (INWARD) DIFFUSION (OF θ-VORTICITY) MUST BE COMPARABLE TO AXIAL ADVECTION (OF θ-VORTICITY). THUS

$$\underbrace{u \frac{\partial \omega_\theta}{\partial z}}_{\substack{\text{AXIAL ADVECTION OF} \\ \theta\text{-VORTICITY}}} \approx \underbrace{\nu \frac{\partial^2 \omega_\theta}{\partial r^2}}_{\substack{\text{radial diffusion} \\ \text{of } \theta\text{-vorticity}}}$$

scaling terms $\Rightarrow$ $\frac{u_0}{z} \approx \frac{\nu}{\delta^2}$

solving for $\delta(z)$ $\Rightarrow$ $\boxed{\delta \approx \sqrt{\frac{\nu z}{u_0}} = \frac{z}{Re_z^{1/2}}}$

OR SETTING $\delta = a \Rightarrow$ $z \approx \frac{a^2 u_0}{\nu}$

THUS, FOR A TYPICAL WATER JET OF RADIUS $a = 5mm$ TRAVELING AT SAY $U_0 = 1 m/s$,

$$z \simeq \frac{25(10^{-6})(1)}{10^{-7}} \simeq 250\ m.$$

IN OTHER WORDS, THE VORTICITY CREATED AT THE TAP WILL REMAIN CONFINED TO THIN CIRCUMFERENTIAL BOUNDARY LAYER -- WE CAN NEGLECT THE EFFECT OF VORTICITY (AND THUS VISCOSITY IN OUR ANALYSIS).

CONSIDERING ASSUMED BASIC FLOW $\underline{u} = U_0 \hat{e}_z$ IT IS EASILY CONFIRMED THAT MOMENTUM EQN AND CONTINUITY ARE SATISFIED BY THIS VELOCITY FIELD.

CONSISTENT W/ THE ASSUMPTION (WHICH HASN'T BEEN EXPLICITLY STATED) THAT THE LIQUID JET FLOWS IN A LOW DENSITY, LOW VISCOSITY FLUID (SUCH AS AIR), $\underline{u} = U_0 \hat{e}_z$ IMPLIES THAT THE SHEAR STRESS INDUCED BY THE SURROUNDING FLUID IS ZERO (SINCE $\frac{\partial u}{\partial r} = 0$).
IN REALITY, THIS IS NOT STRICTLY TRUE (SINCE, e.g., $\mu_{air} \neq 0$). HOWEVER, FOR PURPOSES OF DEFINING A SIMPLE BASIC FLOW WHICH CAPTURES THE ACTUAL FLOW

WITHIN THE JET, THIS IS A REASONABLE APPROXIMATION. (AGAIN, OUR APPROXIMATIONS AND ASSUMPTIONS MUST EITHER BE RIGOROUSLY JUSTIFIED THROUGH CLEAR UNASSAILABLE PHYSICAL ARGUMENTS OR THROUGH EXPERIMENTS.)

7) NOTE, SINCE WE ARE ASSUMING $u_0 =$ CONSTANT, WE ARE IMPLICITLY ASSUMING THAT CHANGES IN GRAVITATIONAL POTENTIAL ENERGY ARE SMALL COMPARED TO THE JET'S CHARACTERISTIC KINETIC ENERGY. IN OTHER WORDS, USING BERNOULLI'S EQN (WHICH APPLIES TO INVISCID FLOW), WE CAN DETERMINE A CONDITION ON THE MAXIMUM ALLOWABLE JET LENGTH, Δz_{max}, OVER WHICH U IS ESSENTIALLY CONSTANT:

$$\frac{u_0^2}{2} + g z_1 = \frac{(u_0 + \Delta u)^2}{2} + g z_2$$

(NOTE P W/IN UNDISTURBED JET IS CONSTANT)

$$\Rightarrow \quad z_1 - z_2 = \Delta z_{max} = \frac{1}{2g} u_0^2 \left[1 + 2\left(\frac{\Delta u}{u_0}\right) + \left(\frac{\Delta u}{u_0}\right)^2 - 1 \right]$$

$$= \frac{u_0^2}{2g}\left[2\epsilon + \epsilon^2\right] \qquad \epsilon = \frac{\Delta u}{u_0}$$

FOR $\epsilon \ll 1 \Rightarrow \epsilon \approx \frac{g}{u_0^2}\Delta z \ll 1 \Rightarrow \boxed{\Delta z \ll \frac{u_0^2}{g}}$

THUS, FOR A JET HAVING CHARACTERISTIC SPEED $u_0 = 1\,m/s$, $\Delta z \ll \sim 0.1\,m = 10\,cm \approx 4\,in$. BASED ON ATTACHED PHOTOS (SCALE IS IN UNITS OF CM) THIS CONDITION APPEARS TO BE WELL MET.

GOVERNING EQNS

(1) MOMENTUM $\boxed{\rho \frac{D\underline{u}}{Dt} = -\nabla P}$ (neglecting gravity)

(2) CONTIN $\boxed{\nabla \cdot \underline{u} = 0}$

KINEMATIC CONDITION (PARTICLES ON SURFACE OF JET REMAIN THERE);

$$r = F(z, \theta, t)$$

$$\Rightarrow \frac{dr}{dt} = F_{,z}\frac{dz}{dt} + F_{,\theta}\frac{d\theta}{dt} + F_{,t}$$

(3) $$\Rightarrow \boxed{u = wF_{,z} + V\frac{F_{,\theta}}{r} + F_{,t}} \quad \boxed{\text{on } r = F}$$

$u =$ r-vel.
$V =$ θ-vel.
$w =$ z-vel.

NOTE $V = r\frac{d\theta}{dt}$

NORMAL STRESS CONDITION AT JET/GAS INTERFACE

$$\sigma_{ij}^{(l)} n_j n_i - \sigma_{ij}^{(g)} n_j n_i = \gamma \nabla \cdot \underline{n}$$

$$p^{(l)} \delta_{ij} n_i n_j - p^{(g)} \delta_{ij} n_i n_j = ''$$

(4) $$\boxed{p^{(l)} - p^{(g)} = \gamma \nabla \cdot \underline{n}}$$

Recall that this is the Young-Laplace eqn. where $p^{(l)}$ is the liquid-side pressure, $p^{(g)}$ is the gas-side pressure, γ is the surface tension coefficient, $\underline{n}$ is the outward unit normal to the jet, and $\nabla \cdot \underline{n}$ is the jet's curvature.

NOTE 1: SINCE WE'RE NEGLECTING VISCOSITY, THE VISCOUS TERMS IN THE CONSTITUTIVE RELN. FOR A NEWTONIAN FLUID ($\sigma_{ij} = -p\delta_{ij} + \mu(u_{i,j} + u_{j,i}) - \frac{2}{3}\mu \nabla \cdot \underline{u}\, \delta_{ij}$) ARE NEGLECTED.

NOTE 2: SINCE FLOW IS INVISCID, WE CANNOT IMPOSE A SHEAR STRESS B.C. ON THE JET-GAS INTERFACE (SINCE ORDER OF MOMENTUM EQN GOES

FROM 2 TO 1, i.e., HIGHEST ORDER DERIVATIVES ARE 1ST ORDER IN EQN. (1)).

NOTE 3: ALTHOUGH IT IS POSSIBLE TO USE BERNOULLI'S EQN TO RELATE $p^{(2)}$ TO u (OR EQUIVALENTLY THE VELOCITY POTENTIAL ϕ) WITHIN THE JET (AS WAS DONE IN ANALYZING THE KELVIN HELMHOLTZ INSTABILITY), THIS TURNS OUT TO BE UNNECESSARY IN THIS PROBLEM.

NOTE 4: IF WE NEGLECT SURFACE TENSION EFFECTS, i.e., ASSUME THAT $p^{(2)} = p^{(9)}$ IN (4), WE ESSENTIALLY RETURN TO THE KELVIN-HELMHOLTZ INSTABILITY PROBLEM. THE CONDITION IN EQN. (4) IS CLEARLY CRITICAL TO ANALYZING CAPILLARY INSTABILITY.

NOW SINCE u_0 IS CONSTANT, WE CAN SIMPLIFY THE ANALYSIS BY USING A FRAME OF REFERENCE ATTACHED TO THE MOVING JET, i.e. DO A GALILEAN TRANSFORMATION. SINCE THE FRAME TRANSLATES AT FIXED

VELOCITY (u_0), THE FORCES OPERATING IN THE LAB FRAME ARE IDENTICAL TO THOSE IN THE MOVING (BUT NON ACCELERATING) FRAME. THUS, LET THE BASIC STATE BE GIVEN BY

$$\boxed{\underline{u} = \underline{0}} \qquad (5)$$

ALSO LET $\boxed{p(q) = 0}$ (6) (EQUIV. TO REFERENCING $p(q)$ TO AMBIENT p)

LINEARIZATION (NEGLECT PRODUCTS OF PERTURBATION QTYS.)

LET

$$\boxed{\begin{aligned} \underline{u} &= \cancel{\underline{u}}^{\,0} + \underline{u}' \\ P &= P + p' = \rho(q) + \rho'(q) \\ F &= a + F' \end{aligned}} \qquad (7)$$

WHERE PRIMED QTYS ARE PERTURBATIONS ON THE BASIC STATE VARIABLES. (WE'LL SIMPLIFY NOTATION BY WRITING $\rho(q)$ AS ρ AND $\rho(q)'$ AS ρ'.)

NOW WRITE MOMENTUM EQN. AND LINEARIZE:

$$\rho\left(\underline{u}'_{,t} + u'\underline{u}'_{,r} + \frac{v'}{r}\underline{u}'_{,\theta} + w'\underline{u}'_{,z}\right) = -\nabla(P + p')$$

$$\Rightarrow \boxed{\rho\,\underline{u}'_{,t} = -\nabla p'} \qquad (8)$$

NOTE, $\rho = \rho(q) = \gamma\nabla\cdot\underline{n} = \dfrac{\gamma}{a}$ = CONSTANT.

CONT'D.

$$\boxed{\nabla \cdot \underline{u}' = 0} \qquad (9)$$

KINEMATIC CONDITION

$$u' = w'(a+F')_{,z} + \frac{v'}{r}(a+F')_{,\theta} + (a+F')_{,t}$$

$$\Rightarrow \boxed{u' = F'_{,t}} \quad \boxed{r = a} \qquad (10)$$

NORMAL STRESS CONDITION

$$(P+P') = \gamma \nabla \cdot \underline{n} \qquad (11)$$

$\underline{n}$:

FOR NON-AXISYMMETRIC DISTURBANCES, IT'S EASIEST TO USE A FORMULA FROM VECTOR CALCULUS. FOR A 3-D SURFACE

$$r = F(z, \theta, t)$$

LET

$$G(r, z, \theta, t) = r - F = 0$$

THEN

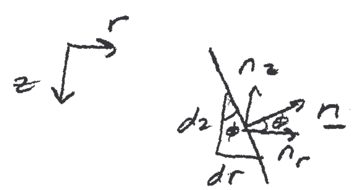
$$\underline{n} = \frac{\nabla G}{|\nabla G|} = \frac{\left(\hat{e}_r \frac{\partial}{\partial r} + \frac{\hat{e}_\theta}{r}\frac{\partial}{\partial \theta} + \hat{e}_z \frac{\partial}{\partial z}\right)(r-F)}{|\nabla G|}$$

$$\Rightarrow \quad \nabla G = \hat{e}_r - \frac{F_{1\theta}}{r}\hat{e}_\theta - F_{1z}\hat{e}_z$$

$$\Rightarrow \quad |\nabla G| = \sqrt{\nabla G \cdot \nabla G} = \sqrt{1 + \left(\frac{F_{1\theta}}{r}\right)^2 + F_{1z}^2}$$

$$\Rightarrow \quad \boxed{\underline{n} = \frac{\hat{e}_r - \frac{F_{1\theta}}{r}\hat{e}_\theta - F_{1z}\hat{e}_z}{\sqrt{1 + \left(\frac{F_{1\theta}}{r}\right)^2 + F_{1z}^2}}} \qquad (12)$$

CHECK USING AXISYMMETRIC DISTURBANCE (WHERE $F_{1\theta} = 0$):



$$n_r = \cos\phi = \frac{dz}{dz\left(1 + \left(\frac{dr}{dz}\right)^2\right)^{1/2}} = \frac{1}{\left[1 + \left(\frac{dr}{dz}\right)^2\right]^{1/2}} = \frac{1}{\sqrt{1 + (F_{1z})^2}}$$

$$n_z = -\sin\phi = \frac{-(dr/dz)}{\left[1 + \left(\frac{dr}{dz}\right)^2\right]^{1/2}} = -\frac{F_{1z}}{\sqrt{1 + F_{1z}^2}}$$

THUS, THIS IS CONSISTENT W/ (11). LIKEWISE, IN CASE WHERE JET IS PERFECTLY CYLINDRICAL, $r = a$; SO THAT $F_{1z} = F_{1\theta} = 0$ AND $\underline{n} = \hat{e}_r$.

NOW WE CAN CALCULATE $\nabla \cdot \underline{n}$:

SINCE JET SURFACE IS DESCRIBED BY $r = F(z,\theta,t)$, WE DON'T INCLUDE $\hat{e}_r \frac{\partial}{\partial r}$ IN THE DIVERGENCE OPERATOR. THUS,

$$\nabla \cdot \underline{n} = \left(\frac{\hat{e}_\theta}{r}\frac{\partial}{\partial\theta} + \hat{e}_z\frac{\partial}{\partial z}\right)\cdot\left(\frac{1}{N}\right)\left[\hat{e}_r - \frac{F_{,\theta}}{r}\hat{e}_\theta - F_{,z}\hat{e}_z\right]$$

WHERE $N = \left[1 + \frac{F_{,\theta}^2}{r^2} + + F_{,z}^2\right]^{1/2}$

$$\nabla \cdot \underline{n} = \left(\frac{1}{N}\right)\left(\frac{1}{r}\right) - \frac{1}{r^2}\frac{\partial}{\partial\theta}\left(\frac{F_{,\theta}}{N}\right) - \frac{\partial}{\partial z}\left(\frac{F_{,z}}{N}\right)$$

$$= \left(\frac{1}{N}\right)\left(\frac{1}{r}\right) - \frac{1}{r^2}\frac{F_{,\theta\theta}}{N} + \frac{1}{r}\frac{F_{,\theta}}{N^3}\left(\frac{F_{,\theta}}{r^2}F_{,\theta\theta} + F_{,z}F_{,z\theta}\right) \qquad (13)$$

$$- \frac{F_{,zz}}{N} + \frac{F_{,z}}{N^3}\left(F_{,\theta}F_{,\theta z} + F_{,z}F_{,zz}\right)$$

NOW LINEARIZE $\nabla\cdot\underline{n}$:

$$\frac{1}{N} = \left[1 + \frac{(F'_{,\theta})^2}{(a+F')^2} + (F'_{,z})^2\right]^{-1/2}$$

$$= \left[1 + \frac{(F'_{,\theta})^2}{a^2}\left(1 - 2\left(\frac{F'}{a}\right)\right) + \left(F'_{,z}\right)^2\right]^{-1/2}$$

LET $H = H\left[(F'_{,\theta})^2, (F'_{,z})^2, F'\right] = H[A,B,C] = \frac{1}{N}$

$$\Rightarrow H = H(0,0,0) + \left.\frac{\partial H}{\partial A}\right|_{0,0,0} A + \frac{\partial H}{\partial B}B + \frac{\partial H}{\partial C}C$$

$$\Rightarrow \frac{1}{N} = 1 - \frac{1}{2}\left(\frac{1}{a^2}\right)(F'_{,\theta})^2 - \frac{1}{2}(F'_{,z})^2$$

LIKEWISE

$$\frac{1}{r} = \frac{1}{a+F'} = \frac{1}{a}\left(1 - \frac{F'}{a}\right)$$

NOW LINEARIZE TERMS IN (13):

$$\Rightarrow \frac{1}{N}\left(\frac{1}{r}\right) = (1)\left(\frac{1}{a}\right)\left(1 - \frac{F'}{a}\right)$$

$$-\frac{1}{r^2}\frac{F_{,\theta\theta}}{N} = -\left(\frac{1}{a^2}\right)(1)\left(F'_{,\theta\theta}\right)$$

$$-\frac{F_{,zz}}{N} = -F'_{,zz}$$

$$\Rightarrow \quad \nabla\cdot\underline{n} = \frac{1}{a} - \left(\frac{F'}{a^2} + \frac{F'_{,\theta\theta}}{a^2} + F'_{,zz}\right)$$

$\Rightarrow$ (11) BECOMES

$$P + p' = \gamma\left[\frac{1}{a} - \left(\frac{F'}{a^2} + \frac{F'_{,\theta\theta}}{a^2} + F'_{,zz}\right)\right]$$

BUT IN BASIC FLOW, THE NORMAL STRESS CONDITION YIELDS

$$P^{(l)} - \cancel{P^{(g)}}^{\,0} = P = \gamma\nabla\cdot\underline{n} = \frac{\gamma}{a} \Rightarrow P = \frac{\gamma}{a}$$

THUS, LINEARIZED NORMAL STRESS CONDITION IS:

$$\boxed{p' = -\gamma\left[\frac{F'}{a^2} + \frac{F'_{,\theta\theta}}{a^2} + F'_{,zz}\right]} \qquad (14)$$

$$\boxed{\text{on } r = a}$$

(14)

THUS, FROM (8), (9), (10), AND (14) WE HAVE THE FOLLOWING LINEARIZED SET OF EQNS. THAT GOVERN THE JET'S STABILITY

$$\rho \underline{u}'_{,t} = -\nabla p' \qquad (8)$$

$$\nabla \cdot \underline{u}' = 0 \qquad (9)$$

$$u' = F'_{,t} \quad (r=a) \qquad (10)$$

$$p' = -\gamma\left[\frac{F'}{a^2} + \frac{F'_{,\theta\theta}}{a^2} + F'_{,zz}\right]_{(r=a)} \qquad (14)$$

NOTE u' = r-COMP. OF PERTURBATION VELOCITY

NORMAL MODES – ASSUME PERTURBATIONS TO THE BASIC FLOW ARE OF NORMAL MODE FORM;

(15) $$(\underline{u}', p', F') = (\hat{\underline{u}}(r), \hat{p}(r), \hat{F})\, e^{st} e^{i(kz + n\theta)}$$

NOTE AZIMUTHAL WAVE MODES MUST BE OF THE FORM $n\theta$ WHERE $n = 0, 1, 2, 3 \ldots$ IN ORDER FOR $\underline{u}'$, p', AND F' AT θ TO BE IDENTICAL TO $\underline{u}'$, p', AND F' AT $\theta + 2\pi$.

NOW TAKE DIVERGENCE OF (8) TO OBTAIN

$$-\rho\,\nabla(\underline{u}'_{,t}) = -\rho\,(\overset{0}{\nabla\cdot\underline{u}'})_{,t} = \boxed{\nabla^2 p' = 0} \qquad (16)$$

VS. (15) → (16) ⇒ $$\frac{d^2\hat{p}}{dr^2} + \frac{1}{r}\frac{d\hat{p}}{dr} - \frac{\hat{p}}{r^2}n^2 - \hat{p}\,k^2 = 0$$

$$\boxed{r^2\hat{p}'' + r\hat{p}' - \hat{p}(n^2 + r^2k^2) = 0} \qquad (17)$$

THIS IS A MODIFIED BESSEL EQN. OF ORDER n.

THE SOLN IS GIVEN BY

$$\hat{p}(r) = a_1 I_n(kr) + b_1 K_n(kr) \qquad (18)$$

(I_n = MODIFIED BESSEL FN. OF FIRST KIND)

SINCE $K_n \to \infty$ AS $r \to 0$ THEN $b_1 = 0$

$$\Rightarrow \boxed{\hat{p}(r) = a_1 I_n(kr)} \qquad (19)$$

NOW INTEGRATING (8) W.R.T. TIME yields

$$\underline{u}'(r,z,\theta,t) - \underline{u}'(r,z,\theta,0) = -\frac{1}{\rho}\int_0^t \left[\frac{d\hat{p}}{dr}\hat{e}_r + \frac{1}{r}\hat{p}\,in\,\hat{e}_\theta + \hat{p}\,ik\,\hat{e}_z\right]e^{i(kz+n\theta)}\cdot e^{st}dt$$

$$= -\frac{1}{\rho s}\quad e^{i(kz+n\theta)}\left[\frac{d\hat{p}}{dr},\ \frac{in\hat{p}}{r},\ ik\hat{p}\right](e^{st}-1)$$

BUT, SINCE $\underline{u}'(r,z,\theta,0) = \hat{\underline{u}}(r)e^{i(kz+n\theta)}$

$\underline{u}'(r,z,\theta,t) = \hat{\underline{u}}(r)e^{i(kz+n\theta)}e^{st}$

THEN

$$\underline{\hat{u}}(r) = -\frac{1}{\rho s}\left[\frac{d\hat{P}}{dr}, \frac{in\hat{P}}{r}, ik\hat{P}\right] \qquad (20)$$

OR USING (19)

$$(21) \qquad \boxed{\underline{\hat{u}}(r) = -\frac{a_1}{\rho s}\left[k I_n'(kr), \frac{in}{r} I_n(kr), ik I_n(kr)\right]}$$

r-comp θ-comp z-comp

NOW FROM (21) AND (10) WE GET

$$-\frac{a_1}{\rho s} k I_n'(ka) = s\hat{F}$$

$$\Rightarrow \quad \boxed{\hat{F} = -\frac{a_1}{\rho s^2} k I_n'(ka)} \qquad (22)$$

THUS, (14) YIELDS

$$\hat{P}(a) = \boxed{a_1 I_n(ka) = -\gamma\hat{F}\left[\frac{1}{a^2} - \frac{n^2}{a^2} - k^2\right]} \qquad (23)$$

ELIMINATING a_1 & $\hat{F}$ FROM (22) & (23) YIELDS THE EIGENVALUE RELATIONSHIP FOR THE GROWTH RATE s:

$$I_n(ka) = \frac{\gamma k}{\rho s^2} I_n'(ka)\left[\frac{1}{a^2} - \frac{n^2}{a^2} - k^2\right]$$

$$\Rightarrow \quad \boxed{s^2 = \frac{\gamma k}{\rho a^2}\frac{I_n'(ka)}{I_n(ka)}\left[1 - n^2 - a^2k^2\right]} \quad (29)$$

LET $\beta = ka \overset{=}{\cdot}$ $2\pi a/\lambda$. THEN (29) CAN BE REWRITTEN AS

$$\boxed{s^2 = \frac{\gamma \beta}{\rho a^3}\frac{I_n'(\beta)}{I_n(\beta)}\left[1 - n^2 - \beta^2\right]} \quad (30)$$

THE STABILITY CHARACTERISTICS OF THE CYLINDRICAL JET CAN NOW BE DETERMINED BY ANALYZING (30) (OR (29)). FIRST NOTE THAT

$$\frac{I_n'(\beta)}{I_n(\beta)} > 0 \quad \text{FOR ALL } \beta > 0.$$

($\beta > 0$ SINCE $\beta = 0$ IMPLIES $\lambda \to \infty$.)

① IN THE CASE WHERE $\boxed{n \neq 0}$, $1 - n^2 - \beta^2 < 0$. THUS, $s^2 < 0$ SO THAT $s = \pm i\omega \Rightarrow$ ALL SUCH MODES ARE THUS STABLE $(\sigma = 0)$.

NOTE THAT EACH MODE IN THIS CASE PERSISTS AS A STABLE CYCLIC OSCILLATION OF FREQUENCY

$$\omega = \left(\frac{\gamma\beta}{\rho a^3}\frac{I_n'(\beta)}{I_n(\beta)}\left[n^2 + \beta^2 - 1\right]\right)^{1/2}.$$

THUS, FOR $n \neq 0$ JET IS STABLE TO ALL NON-AXISYMMETRIC MODES.

② IN THE CASE WHERE $\boxed{n = 0}$, THE JET IS UNSTABLE IF $1 - \beta^2 > 0$, i.e., IF $\boxed{0 < \beta < 1}$.

IN THE CASE $n = 0$, PERTURBATIONS HAVE NO θ-DEPENDENCE -- THESE ARE SYMMETRIC MODES. FOR A GIVEN JET RADIUS a, THE RANGE OF UNSTABLE WAVELENGTHS (FROM THE CONDITION $0 < \beta < 1$) SATISFIES

$$0 < \frac{2\pi a}{\lambda} < 1$$

OR

$$0 < 2\pi a < \lambda$$

THUS, ANY AXISYMMETRIC DISTURBANCE (i.e., MODE) HAVING A WAVELENGTH GREATER THAN THE JET'S CIRCUMFERENCE IS UNSTABLE.

IN CONTRAST, ANY AXISYMMETRIC MODE HAVING WAVELENGTH LESS THAN $2\pi a$ IS STABLE.

AS SHOWN ON THE ATTACHED GRAPH, COMPARING EXPERIMENTAL AND THEORETICAL CURVES OF $s\sqrt{\frac{\rho a^3}{\gamma\beta}}$ $\left(= \frac{I_0'(\beta)}{I_0(\beta)}[1-\beta^2]\right.$ FROM (30)$\left.\right)$ VS. β, THE LINEAR THEORY PREDICTS WELL OBSERVED VALUES OF $s\sqrt{\frac{\rho a^3}{\gamma\beta}}$.

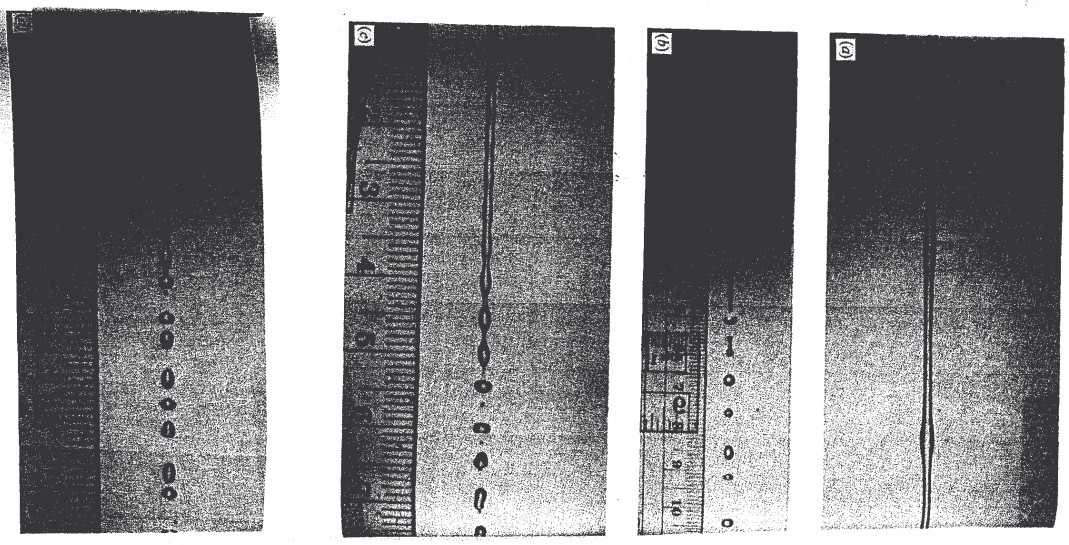
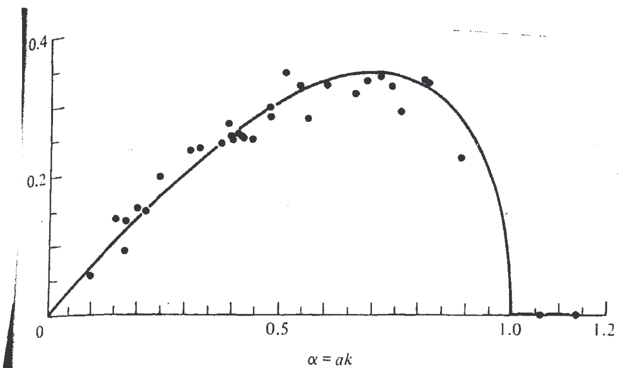


Fig. 1.6. The break-up of a jet of water, cf. Fig. 1.5. (a) $\alpha = 0.148$. (b) $\alpha = 0.262$. (c) $\alpha = 0.678$. (d) $\alpha = 0.805$. (From Donnelly & Glaberson 1966.)



ig. 1.5. The growth rates of unstable capillary modes of a uniform jet of uid. (The solid line is based on the solution of equation (5.17) for an viscid fluid by Chandrasekhar (1961), p.538 and the experimental points e from Donnelly & Glaberson (1966).)