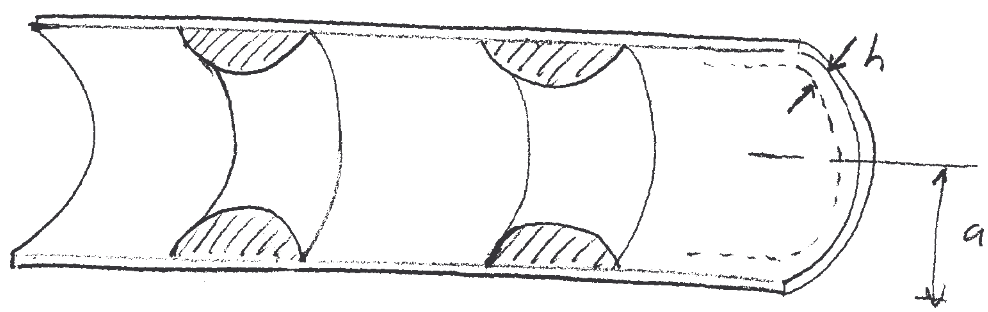
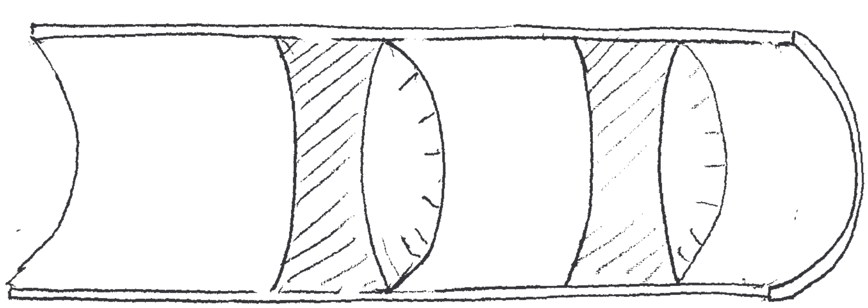
EXAMPLE II — A THIN ANNULAR FILM OF FLUID COATS THE INSIDE OF A CLEAN GLASS TUBE. IF AIR IS BLOWN GENTLY DOWN THE TUBE, IT IS OBSERVED THAT THE ANNULAR FILM DESTABILIZES DUE TO SURFACE TENSION AND BREAKS TO FORM EITHER COLLARS OR LENSES, DEPENDING ON THE RATIO OF INITIAL FILM THICKNESS, $h$, TO TUBE RADIUS, $a$. WE WILL ANALYZE THIS USING




COLLARS



LENSES

SOLN

HERE, THE NORMAL STRESS BALANCE ACROSS THE LIQUID/AIR INTERFACE IS GIVEN BY

$$\boxed{\Delta P = P_{OUT} - P_{IN} = \sigma \nabla_H \cdot \hat{n}} \quad (2.1)$$

WHERE

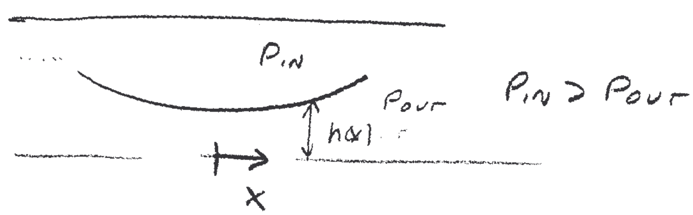
$$\boxed{\nabla_H \cdot \hat{n} = \frac{1}{h(1+(h')^2)^{1/2}} - \frac{h''}{(1+(h')^2)^{3/2}}} \quad (2.2)$$

WAS OBTAINED IN EQN. (10) OF 1st EXAMPLE.

$\boxed{\text{ASIDE}}$

QUICK CHECK THAT SIGN ON $\Delta P$ IS CORRECT;

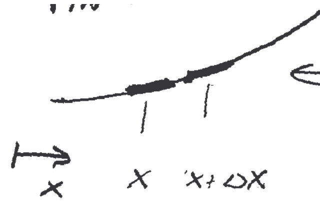
IN THE CASE WHERE $P_{OUT} - P_{IN} < 0$, THE SURFACE WILL APPEAR AS FOLLOWS:




WHEN $h(x)$ IS LARGE, $\nabla_H \cdot \hat{n} \to \frac{-h''}{(1+(h')^2}$

NOW, CONSIDERING TWO ADJACENT ELEMENTS OF SURFACE FOR $x > 0$, WE SEE THAT WHEN $P_{IN} - P_{OUT}$ (USING THE ABOVE PICTURE

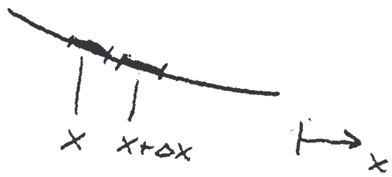
$$h'' \approx \frac{h'(x+\Delta x) - h'(x)}{\Delta x} > 0$$




SINCE $h'(x+\Delta x) > h'(x)$

THEN $h''(x) > 0$

$P_{IN} > P_{OUT}$




LIKEWISE, FOR $x < 0$,
SINCE $h'(x+\Delta x) > h'(x)$
($h'(x+\Delta x)$ IS <u>LESS</u> NEGATIVE THAN $h'(x)$)
THEN $h''(x) \approx \dfrac{h'(x+\Delta x) - h'(x)}{\Delta x} > 0$

THUS, FROM PHYSICAL CONSIDERATIONS, WHEN $P_{IN} > P_{OUT}$, $\underline{h'' > 0}$ for all $x$. REFERRING TO (2.1) AND (2.2), WE SEE THAT THIS IS FULLY CONSISTENT W/ THE NORMAL STRESS BALANCE (IN THE CASE WHERE $h$ IS LARGE). THUS, THE SIGN ON $\Delta P$ IN (2.1) IS CORRECT.

END ASIDE

WE WILL SIMPLIFY THE PROBLEM BY ASSUMING THAT $P_{IN}$ = CONSTANT. SINCE THERE'S NO MOTION IN THE LIQUID FILM, IT IS READILY SHOWN (SEE APPENDIX TO PROBLEM) THAT THIS IS A VALID ASSUMPTION WHEN

$$\frac{\sigma}{\rho g a^2} \gg 1 \qquad \text{(HORIZONTAL TUBE)}$$

$\overline{\rho g a L}$ ≫ 1 (VERTICAL PULSE)

WHERE $L$ IS THE CHARACTERISTIC AXIAL LENGTH OF THE LENS OR COLLAR.

THUS, NORMAL STRESS BALANCE (YOUNG-LAPLACE EQN) IS GIVEN BY

$$\boxed{\frac{\Delta P}{\sigma} = \frac{1}{h\,(1+(h')^2)^{1/2}} - \frac{h''}{(1+(h')^2)^{3/2}}} \qquad (2.3)$$

USING THE SAME PROCEDURE AS IN FIRST EXAMPLE, WE GET

$$f^2 = 1 + (h')^2$$

$$\Rightarrow f^{-1} = (1 + (h')^2)^{-1/2}$$

$$= \frac{dx/dh}{\left(1 + \left(\frac{dx}{dh}\right)^2\right)^{1/2}} \qquad \text{(SEE PG. 12)}$$

THUS, AS BEFORE (SEE EQNS. (13) & (14))

$$\boxed{\frac{f^{-1}}{h} + \frac{d}{dh}(f^{-1}) = \frac{\Delta P}{\sigma} \equiv P_0} \qquad (2.4)$$

INTEGRATING FACTOR $= e^{\int \frac{1}{h} dh} = e^{\ln h} = h$

$$\Rightarrow (h)\cdot(2.4) \Rightarrow \quad h(f^{-1})' + f^{-1} = P_0 h$$

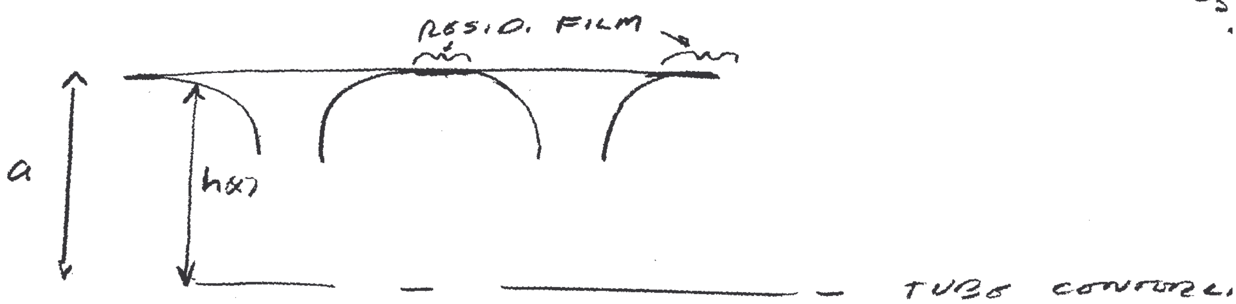
$$\Rightarrow \quad (h f^{-1})' = P_0 h$$

$$\Rightarrow \quad f^{-1} = \frac{P_0}{2} h + \frac{C_0}{h} \qquad (C_0 = \text{integ. const.})$$

OR

$$(1+(h')^2)^{-1/2} = \frac{P_0}{2} h + \frac{C_0}{h} \qquad (2.5)$$

IN ORDER TO DETERMINE $C_0$, LET'S ASSUME THAT THE CONTACT ANGLE BETWEEN THE LIQUID AND TUBE WALL IS 0. THIS IS EQUIVALENT TO $h' \to 0$ AS $h \to a$. AN ALTERNATIVE AND MORE REALISTIC ARGUMENT IS THAT AFTER THE ANNULAR FILM HAS BIFURCATED INTO A COLLAR OR LENS, A THIN RESIDUAL LIQUID FILM WILL REMAIN BETWEEN ADJACENT COLLARS/LENSES.




THIS MEANS WE CAN AGAIN ASSUME THAT

$$\boxed{h' \to 0 \quad \text{AS} \quad h \to a} \qquad (2.6)$$

THUS, USING (2.6) IN (2.5) WE OBTAIN

$$C_0 = a - \frac{P_0}{2} a^2$$

$(2.5) \Rightarrow \quad (1+(h')^2)^{-1/2} = \frac{P_0}{2}h + \left(a - \frac{P_0 a^2}{2}\right)/h$

$$\Rightarrow \quad (h')^2 = -1 + \frac{h^2}{\left(\frac{P_0}{2}(h^2 - a^2) + a\right)^2}$$

$$(h')^2 = \frac{h^2 - \left(\frac{P_0}{2}(h^2-a^2)+a\right)^2}{\left(\frac{P_0}{2}(h^2-a^2)+a\right)^2} \qquad (2.7)$$

LET $\quad H = h/a$ = DIMENSIONLESS SURFACE HEIGHT

$G = \frac{P_0 a}{2}$ = DIM'LESS PRESSURE

$X = x/a$ = DIMENSIONLESS AXIAL POSITION

$$(2.7) \Rightarrow \quad \boxed{(H')^2 = \frac{H^2 - (G(H^2-1)+1)^2}{(G(H^2-1)+1)^2}} \qquad (2.8)$$

$$(H')^2 = \frac{H^2 - G^2(H^2-1)^2 - 2G(H^2-1) - 1}{(\quad '' \quad)}$$

$$= \frac{(H^2-1)(1 - G^2(H^2-1) - 2G)}{(\quad '' \quad)}$$

$$= \frac{(H^2-1)(-G^2H^2 + G^2 - 2G + 1)}{(\quad '' \quad)}$$

$$(H')^2 = \frac{(H^2-1)\left(-G^2H^2 + (1-G)^2\right)}{\left(G(H^2-1)+1\right)^2}$$

$$\boxed{(H')^2 = \frac{(1-H^2)\left(G^2H^2 - (1-G)^2\right)}{\left(G(H^2-1)+1\right)^2}} \quad \text{✱} \quad (2.9)$$

CASE 1 $\boxed{G > 1}$

SINCE $(H')^2 \geq 0$ AND SINCE $H \leq 1$

AND DENOMINATOR $\geq 0$, THEN

$$G^2H^2 - (1-G)^2 \geq 0$$

$$\Rightarrow \quad H^2 \geq \frac{(1-G)^2}{G^2}$$

$$H \geq -\left(\frac{1-G}{G}\right) = \frac{G-1}{G}$$ (choose "$-$" sign for SQ RT. IN ORDER FOR $H>0$

$$\Rightarrow \quad \boxed{1 \geq H \geq \frac{G-1}{G}}$$

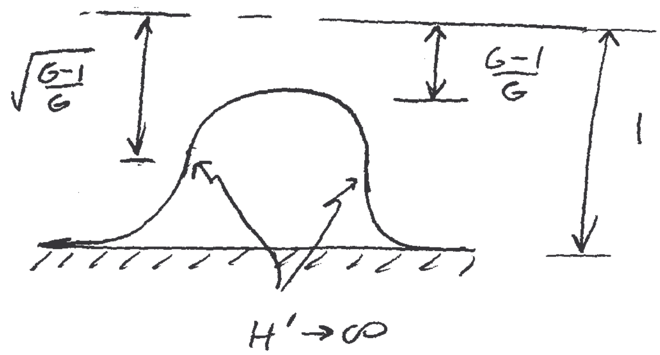
ALSO SINCE DENOMINATOR IN $(2.9) = 0$

WHEN $H^2 = \frac{G-1}{G}$ ; IE, WHEN $H = \sqrt{\frac{G-1}{G}}$

$H' \to \infty$. THUS, THE APPROXIMATE

SHAPE OF THE INTERFACE IS AS

FOLLOWS




$\boxed{\text{CASE 1 SOLN}}$

CASE 2 $\boxed{P=1}$ $\Longrightarrow$ LENSES

(2.9) BECOMES

$$(H')^2 = \frac{1-H^2}{H^2}$$

$$HH' = \pm(1-H^2)^{1/2}$$

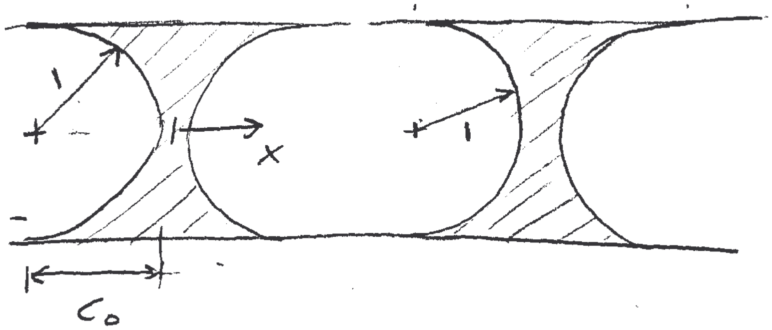
$$\Rightarrow \int \frac{\frac{1}{2}\,d(H^2)}{\pm(1-H^2)^{1/2}} = \int dX$$

$$\Rightarrow \mp(1-H^2)^{1/2} = X + c_0$$

$$1-H^2 = (X+c_0)^2$$

$$\Rightarrow H^2 + (X+c_0)^2 = 1 \Longrightarrow$$

EQN OF A HEMISPHERE W/ CENTER AT $(H=0,\ X=-c_0)$




## LENSES

NOTE SINCE $G = \frac{P_0 a}{2} = \frac{\Delta P a}{2\sigma} = 1$

THEN

$$\boxed{\Delta P = \frac{2\sigma}{a}}.$$ SINCE $\sigma \nabla_H \cdot \hat{n} = \frac{2\sigma}{a}$

FOR A HEMISPHERE, THEN $G = 1$ IS FULLY CONSISTENT W/ THE PREDICTED HEMISPHERICAL SHAPE.

THE SOLNS IN THIS CASE DESCRIBE LENSES.

THE VOLUME OF LIQUID PER WAVELENGTH $L$ IS GIVEN BY

$$V = \pi \cancel{R}^{2} L - 2\left(\tfrac{1}{2}\right)\tfrac{4}{3}\pi \cancel{R}^{3}$$

$$V = \pi\left(L - \tfrac{4}{3}\right)$$

THE MINIMUM VOLUME POSSIBLE OCCURS WHEN ADJACENT HEMISPHERES TOUCH. (WHICH WE'LL FIND IS THE MINIMUM POSSIBLE ON STABILITY GROUNDS). IN THIS CASE, $L = 2$ AND $V = \frac{2\pi}{3}$.

CASE 3 $G < 1$ | COLLARS

FROM (2.9) $(H')^2 \geq 0$, SINCE $(1-H^2) \geq 0$

AND $(G(H^2-1)+1)^2 > 0$, THEN

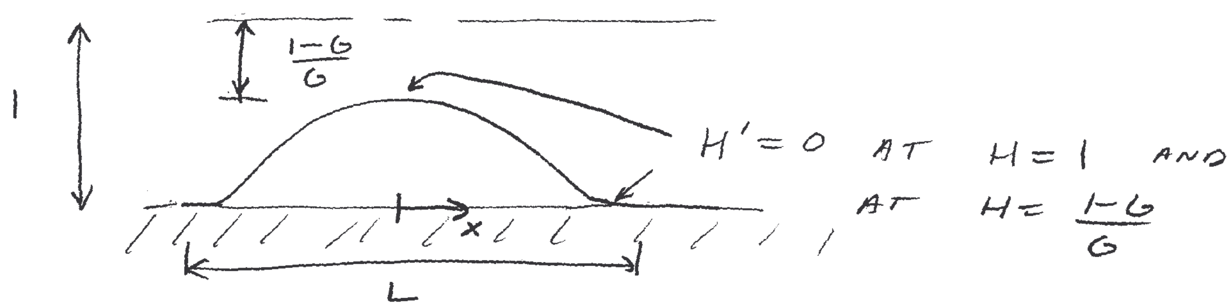
$$G^2H^2 - (1-G)^2 \geq 0$$

$$\Rightarrow \quad \underline{H \geq \frac{1-G}{G}}$$

OR SINCE $H \leq 1$ THEN

$$\boxed{1 \geq H \geq \frac{1-G}{G}}$$

$$\Rightarrow \quad 1 \geq \frac{1-G}{G} \quad \Rightarrow \quad \boxed{G \geq 1/2}$$




NOTE SINCE $H' > 0$ EXCEPT
WHERE $H = \frac{1-G}{G}$ (WHERE $H$ IS A MINIMUM
AND WHERE $H = 1$ ($= H_{max}$), THEN WE
CAN INFER THAT THE SHAPE IS
QUALITIVELY AS SHOWN. A NUMERICAL

SOLN OF (2,9) ...

[NOTE, A NUMERICAL SOLN. WOULD BE BASED ON A 'SHOOTING METHOD' WHERE WE ITERATIVELY GUESS $L$, INTEGRATE (2.9) FROM $X = -L/2$ AND FROM $X = L/2$ TO $X = 0$ UNTIL THE SLOPE OF $H$ AT $X = 0$ IS ZERO.]

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$$\boxed{\text{STABILITY AND BIFURCATION BETWEEN LENSES AND COLLARS}}$$

IN ORDER TO EXAMINE OBSERVED BIFURCATION PHENOMENA, LET'S CALCULATE THE LIQUID VOLUME FOR THE NONDIMENSIONAL PRESSURE RANGE $1 \geq G \geq \frac{1}{2}$.

$$\boxed{\forall(G) = 2\int_0^{L/2} \left(\pi R^2 - \pi H^2\right) d\bar{X}} \tag{2.10}$$

(NOTE, COLLAR IS SYMMETRIC ABOUT $X = 0$; THUS, WE DOUBLE INTEGRAL BETWEEN $X = 0$ AND $X = L$

(SOLN. METHOD II)

SINCE $L$ (WHICH DEPENDS ON $G$) IS UNKNOWN, WE CAN AVOID AN ITERATIVE SOLN. (METHOD 2) BY INTEGRATING W.R.T. $h$:

$$H' = \frac{dH}{dX}$$

$$\Rightarrow \quad dX = dH/H'$$

IN ORDER TO DETERMINE LIMITS OF $H$ CORRESPONDING TO $X = 0$ AND $X = L/2$ WE NOTE THAT AT $X = 0$, $H' = 0$ AND $H = H_{min} = \frac{1-G}{G}$; IN ADDITION, AT $X = L/2$, $H = 1$. THUS, (2.10) BECOMES

$$\forall(G) = 2\pi \int_{\frac{1-G}{G}}^{1} \frac{(1-H^2)\, dH}{(H')}$$

USING (2.9) TO EXPRESS $H'$ IN TERMS OF $H$ AND $G$, WE OBTAIN

$$(2.11) \quad \boxed{\forall(G) = 2\pi \int_{\frac{1-G}{G}}^{1} \frac{(1-H^2)^{1/2}\left[G(H^2-1)+1\right]}{\left[G^2H^2-(1-G)^2\right]^{1/2}}\, dH}$$

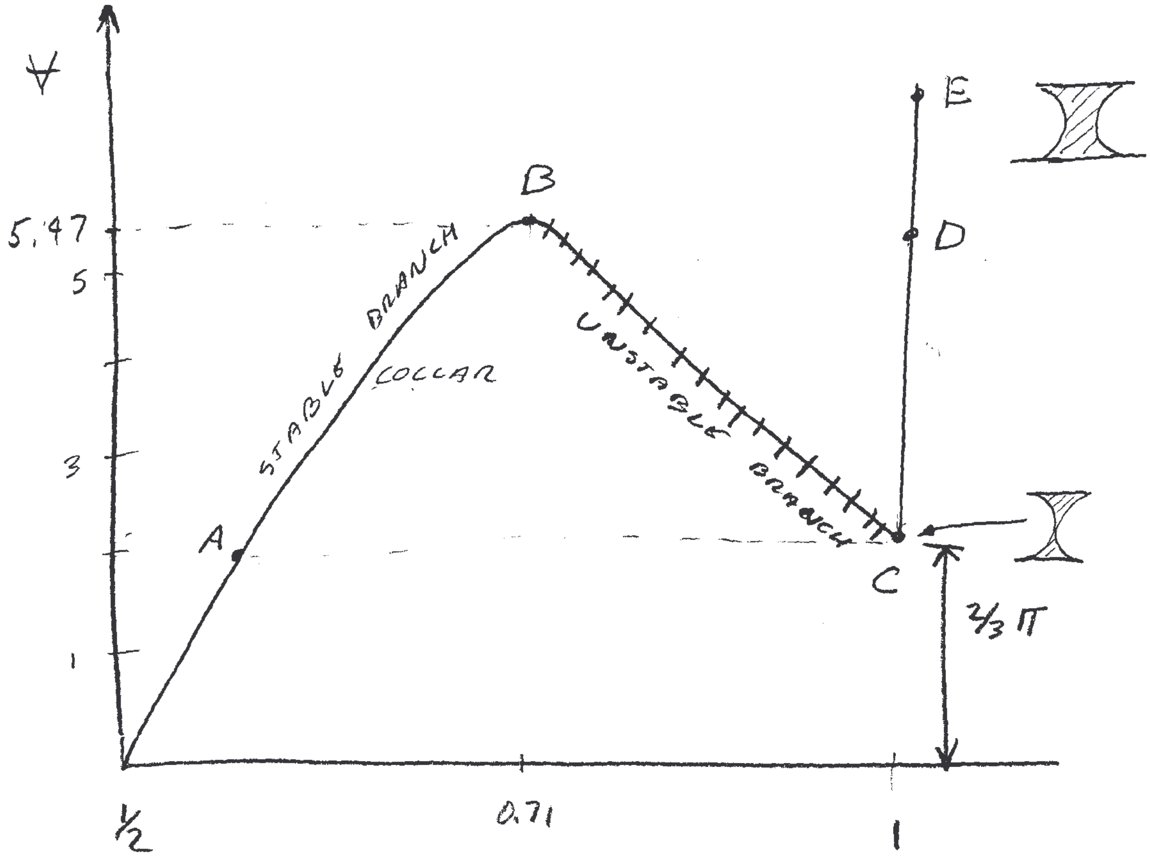
AN INTEGRABLE SINGULARITY EXISTS AT THE LOWER LIMIT OF THE INTEGRAL IN (2.11). WE CAN REMOVE IT USING FOLLOWING SUBSTITUTION

$$H = \frac{1-G}{G}\cosh u \Rightarrow H\Big|_{\frac{1-G}{G}}^{1} \Rightarrow u\Big|_{0}^{\cosh^{-1}\left(\frac{G}{1-G}\right)}$$

$$V(G) = \frac{2\pi}{G}\int_{0}^{\cosh^{-1}\left(\frac{G}{1-G}\right)} \left(1-G+\frac{(1-G)^2}{G}\cosh^2 u\right)\left(1-\frac{(1-G)^2}{G^2}\cosh^2 u\right)^{1/2} du \qquad (2.12)$$

(2.12) CAN BE NUMERICALLY INTEGRATED, see, e.g., PRESS et al., NUMERICAL RECIPES IN FORTRAN (OR C), OR BY COMMERCIAL SOFTWARE (e.g. MATHEMATICA).

THE RESULT GIVES $V$ AS A FN. OF THE DIM'LESS PRESSURE DIFFERENCE, G, ACROSS THE LIQUID INTERFACE.

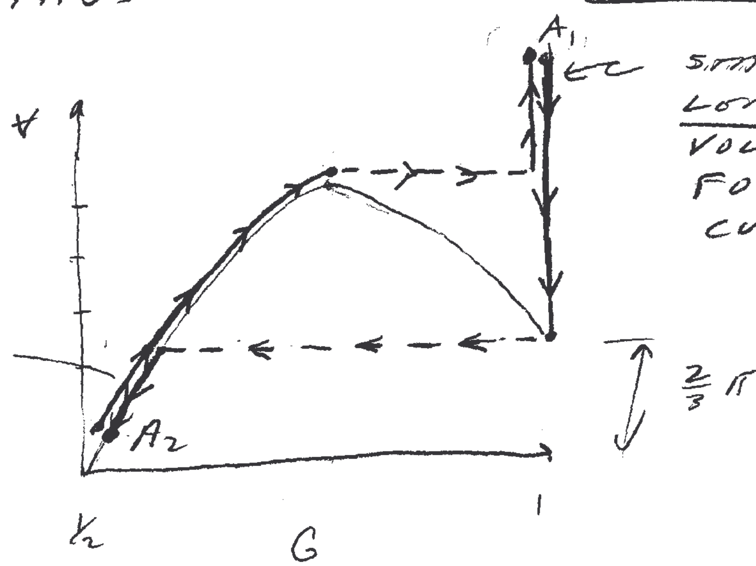



$$G = \frac{a\Delta P}{2\sigma}$$

DISCUSSION – FROM THE NUMERICAL SOLN. TO (2.12), IT IS FOUND THAT THE MAXIMUM COLLAR VOL., $\forall_{c,max}$, OCCURS AT $G = 0.706$ ($\forall_{c,max} = 5.47$).

(THIS VALUE OF $\forall$ CORRESPONDS TO A MINIMUM RADIUS OF 0.42). ANY INCREASE IN COLLAR VOLUME CAUSES THE COLLAR TO BIFURCATE TO POINT D ON THE RESPONSE CURVE,

I.E., THE COLLAR LOSES STABILITY (BIFURCATES) TO A LENS SHAPE AT PT. D. IN CONTRAST, IMAGINE AN EXPERIMENT IN WHICH THE VOLUME OF LIQUID IN A LENS IS INITIALLY THAT AT PT. E. IF WE GRADUALLY REDUCE THE LENS VOLUME UNTIL ADJACENT HEMISPHERICAL SURFACES TOUCH AT PT. C, THEN ANY FURTHER VOLUME REDUCTION CAUSES A BIFURCATION FROM A LENS SHAPE AT C TO A COLLAR SHAPE AT PT. A. THIS SYSTEM THUS DISPLAYS <u>HYSTERESIS</u>:




STARTING FROM <u>LENS</u> SHAPE AT $A_1$, VOLUME <u>REDUCTION</u> FOLLOW CURVE $A_1A_2$.

STARTING FROM <u>COLLAR</u> SHAPE AT $A_2$, VOLUME <u>INCREASES</u> FOLLOW CURVE $A_2A_1$.

NOTE, FOR $\forall < \frac{2}{3}\pi$ <u>ONLY</u> COLLARS ARE STABLE, WHILE FOR $\forall > 5.4$ <u>ONLY</u> LENS ARE STABLE.

FOR $\frac{2}{3}\pi \le V \le 5.07$, BOTH COLLARS AND LENSES ARE STABLE; THE SOLUTION OBSERVED DEPENDS ON INITIAL CONDITIONS.

<u>EXPERIMENT</u>:- FORM A LIQUID FILM INSIDE A STRAW. IF YOU BLOW GENTLY THE FILM IS RELATIVELY THICK AND EVOLVES INTO <u>LENSES</u>. BLOW HARD SO THAT A THIN FILM IS LEFT; THIS WILL EVOLVE INTO <u>COLLARS</u>.