ILLUSTRATION: USE OF A BIFURCATION THEORY FOR INTERPRETING CAPILLARY INSTABILITY

A VERY CURSORY OVERVIEW OF BIFURCATION THEORY

BIFURCATION THEORY IS AN AREA OF STUDY IN NONLINEAR DIFFERENTIAL EQUATIONS AND DYNAMICAL SYSTEMS.

AS A BRIEF INTRODUCTION, A A SINGLE OR SET (SYSTEM) OF NONLINEAR DIFFERENTIAL EQNS. CAN BE EXPRESSED AS

$$\underline{\dot{x}} = \underline{f}(\underline{x}) \qquad (1)$$

WHERE $\underline{X} = (x_1, x_2, x_3 \ldots x_N)$ IS A SET OF N TIME DEPENDENT VARIABLES, $\underline{f} = [f_1(x_1, x_2 \ldots x_N), \ldots f_N(x_1, x_2 \ldots x_N)]$ IS A SET ON N FUNCTIONS AND $\underline{\dot{X}} = (\dot{x}_1, \dot{x}_2, \ldots \dot{x}_N)$. ALTHOUGH IT IS CERTAINLY POSSIBLE TO NUMERICALLY SOLVE THE SYSTEM OF D.E.'S IN (1), IT IS OFTEN MORE INFORMATIVE TO STUDY THE DYNAMICAL (i.e., TIME DEPENDENT) BEHAVIOR OF THE SYSTEM GEOMETRICALLY (i.e., GRAPHICALLY). IN ORDER TO STUDY THE SYSTEM GEOMETRICALLY, A PHASE PORTRAIT OF THE SYSTEM IS TYPICALLY COMPUTED. (USUALLY NUMERICALLY).

A PHASE PORTRAIT PROVIDES A COMPLETE GRAPHICAL DESCRIPTION OF THE NONLINEAR SYSTEM'S (REPRESENTED BY (1)) BEHAVIOR IN TIME. THE ATTACHMENT PROVIDES A FEW SIMPLE EXAMPLES OF BOTH LINEAR AND NONLINEAR DYNAMICAL SYSTEMS AND ASSOCIATED PHASE PORTRAITS. [HISTORICAL NOTE: THE QUALITATIVE OR GEOMETRIC APPROACH TO STUDYING NONLINEAR DIFF'TL. EQNS WAS INITIATED BY HENRI POINCARE IN THE LATE 1800'S / EARLY 1900'S.]

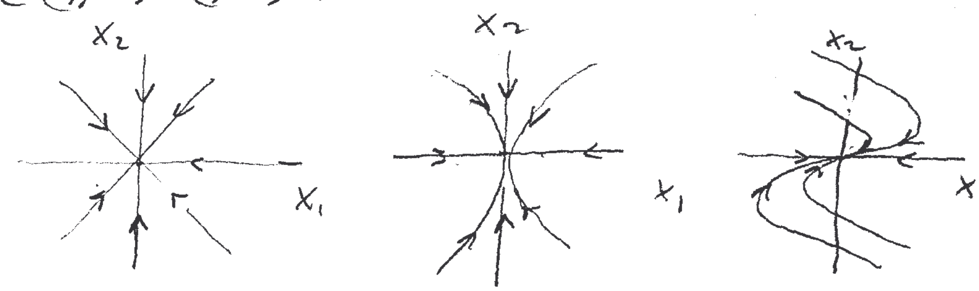
SINCE WE ARE TYPICALLY (BUT NOT EXCLUSIVELY) INTERESTED IN THE LONG-TIME BEHAVIOR OF THE SYSTEM IN (1) (i.e., HOW THE SYSTEM BEHAVES AS t BECOMES LARGE) AND IN THE SYSTEM'S EQUILIBRIUM POINTS (STATES), i.e., WHERE $\dot{\underline{x}} = \underline{0}$, THEN MUCH OF THE GEOMETRIC THEORY FOCUSES ON HOW LINEAR AND NONLINEAR SYSTEMS BEHAVE IN THESE INSTANCES!

THE <u>GLOBAL</u> BEHAVIOR OF A DYNAMICAL SYSTEM, REPRESENTED GRAPHICALLY BY ITS <u>PHASE</u> <u>PORTRAIT</u>, IS GENERALLY CHARACTERIZED BY EXISTANCE OF

<u>CRITICICAL</u> (= <u>FIXED</u> = <u>EQUILIBRIUM</u> = <u>SINGULAR</u>) <u>POINTS</u> AT VARIOUS LOCATIONS ON THE PH. PORT. $[\underline{\dot{X}} = \underline{0}$ IN (1) AT ANY CRIT. PT.$]$

IN 2-DIMENSIONAL SYSTEMS ($N=2$ IN EQS. (1), i.e., $\dot{x}_1 = f_1(x_1, x_2)$, $\dot{x}_2 = f_2(x_1, x_2)$), THE FOLLOWING <u>CRITICAL</u> <u>POINTS</u> ARE POSSIBLE:

<u>NODES</u> ALL @ $(x_1, x_2) = (0,0)$:



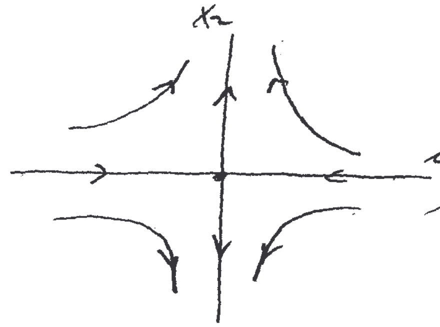
THESE ARE <u>STABLE</u> <u>NODES</u> - ANY SYSTEM TRAJECTORY THAT APPROACHES A <u>STABLE</u> <u>NODE</u> WILL EVENTUALLY BE <u>ATTRACTED</u> TO THE NODE AND WILL REMAIN THEIR INDEFINITELY.

<u>UNSTABLE</u> <u>NODES</u> HAVE THE SAME APPEARANCE AS <u>STABLE</u> <u>NODES</u> EXCEPT THAT SYSTEM TRAJECTORIES

ALL LEAVE (ARE REPULSED BY) THE NODE (i.e., DIRECTION OF ARROWS IN LAST 3 PICTURES ARE REVERSED).

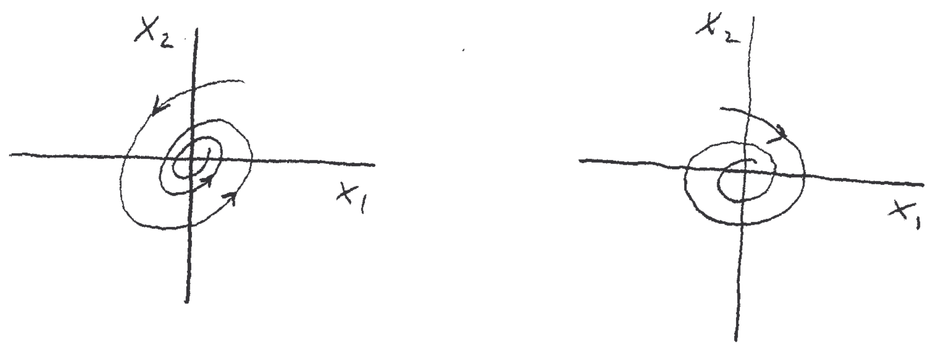
<u>STABLE</u> <u>NODES</u> ALSO CALLED <u>SINKS</u>.
<u>UNSTABLE</u> <u>NODES</u> " " <u>SOURCES</u>.

<u>SADDLE</u> <u>POINTS</u> (@ $X_1, X_2 = 0,0$)



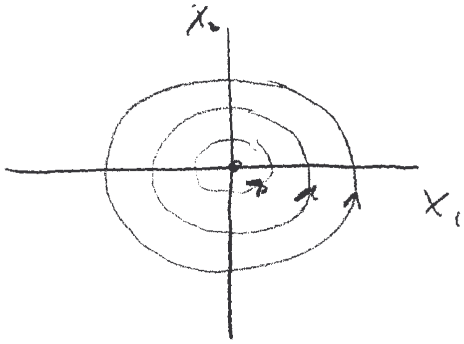
NOTE - TRAJECTORIES WHICH GO DIRECTLY TO OR FROM SADDLE PT. ARE CALLED <u>SEPARATRICES</u>

<u>STABLE</u> <u>FOCUS</u> (@ $X_1, X_2 = 0,0$)



<u>UNSTABLE</u> <u>FOCUS</u> — HAS SAME APPEARANCE AS STABLE FOCUS EXCEPT TRAJECTORIES LEAVE THE FOCUS (i.e., ARROWS REVERSED IN LAST TWO PICS.)

CENTER ie $x_1, x_2 = 0, 0$



TRAJECTORIES NEAR A CENTER WILL REMAIN NEAR THE CENTER AND LOOP AROUND THE CENTER AS SHOWN FOR ALL TIME.

[A NONLINEAR DYNAMICAL SYSTEM'S CRITICAL PTS CAN BE FOUND BY NUMERICAL OR IN SIMPLE CASES, ANALYTICAL, MEANS. ONCE LOCATED, THE NATURE OF EACH CRITICAL PT., i.e., WHETHER A CRITICAL POINT IS A FOCUS, CENTER, SADDLE, NODE (ALL OF WHICH ARE REFERRED TO AS HYPERBOLIC CRIT. PTS.) OR A NON-HYPERBOLIC CRIT. PT., CAN BE DETERMINED BY ANALYZING SYSTEM BEHAVIOR (LOCAL ANALYSIS) NEAR THE POINT. DETAILS ARE GIVEN, e.g., IN PERKO, Chs. 2-3.]

A <u>BIFURCATION</u> OCCURS WHEN A <u>SMALL</u> CHANGE IN $\underline{f}(\underline{x})$ IN THE <u>DYNAMICAL SYSTEM</u> (i.e., SET OF DIFF'L EQNS) GIVEN BY

$$\dot{\underline{x}} = \underline{f}(\underline{x}) \tag{2}$$

RESULTS IN A <u>QUALITATIVE CHANGE</u> IN THE SYSTEM'S PHASE PORTRAIT (i.e., THE NUMBER AND/OR TYPE OF CRITICAL POINTS AND/OR THE NUMBER/TYPE OF <u>ATTRACTING SETS</u>/<u>ATTRACTORS</u> CHANGE IN SOME WAY). See Perko Ch. 4 for a MATHEMATICAL INTRODUCTION.

THE NEXT TWO PAGES GIVE A SIMPLE EXAMPLE OF A 2-D SYSTEM

$$\begin{aligned}\dot{x} &= -y\\ \dot{y} &= x\end{aligned} \Rightarrow \dot{\underline{x}} = \underline{f}(\underline{x})$$

WHICH IS <u>STRUCTURALLY UNSTABLE</u> (i.e., A SMALL Δ IN $\underline{f}(\underline{x})$ RESULTS IN A QUALITATIVE CHANGE IN THE SYSTEM'S PHASE PORTRAIT).

mount of friction, i.e., damping, will change the undamped, periodic mo-ion seen in Figure 1 of Section 2.14 in Chapter 2 to a damped motion; e., the centers in Figure 1 will become stable foci. Of course, a frictionless endulum is not physically realizable. If we were to only consider physi-al problems which lead to systems of differential equations in $\mathbf{R}^2$, then e would not have to worry about arbitrarily small changes in the model eading to qualitatively different behavior of the system. However, there re higher dimensional systems (with $n \geq 3$) which are realistic models for ertain physical problems (such as the three-body problem) and which are tructurally unstable. Recently, many dynamical systems have been found hich model physical problems and which have a strange attractor as part f their dynamics. These systems are not structurally stable and yet they re realistic models for certain physical systems; cf., e.g., [G/H], p. 259.

We next define what is meant by a structurally stable vector field on n n-dimensional compact manifold M: If $\mathbf{f}$ is a C^1-vector field on an n-imensional, compact, differentiable manifold M, then for any (finite) atlas $U_j, \mathbf{h}_j\}_{j=1}^m$ for M we define the *C^1-norm of* $\mathbf{f}$ *on* M as

$$\|\mathbf{f}\|_1 = \max_j \|\mathbf{f}_j\|_1$$

here $\mathbf{f}_j: V_j \to \mathbf{R}^n$ and for $j = 1, \ldots, m$, $V_j = \mathbf{h}_j(U_j) \subset \mathbf{R}^n$ as in Section .10 of Chapter 3. Note that for different atlases for M we will get different orms on $C^1(M)$; however, all of these norms will be equivalent. Recall hat two norms $\|\cdot\|_a$ and $\|\cdot\|_b$ on a linear space L are said to be equivalent f there are positive constants A and B such that

$$A\|\mathbf{x}\|_a \leq \|\mathbf{x}\|_b \leq B\|\mathbf{x}\|_a$$

or all $\mathbf{x} \in L$. Hence the resulting topologies on $C^1(M)$ will be equivalent. Ve say that two C^1-vector fields $\mathbf{f}, \mathbf{g} \in C^1(M)$ are *topologically equivalent* n M if there is a homeomorphism $H: M \to M$ which maps trajectories of 2) on M onto trajectories of (2′) on M and preserves their orientation by ime.

Definition 2. Let $\mathbf{f}$ be a C^1-vector field on a compact, n-dimensional, lifferentiable manifold M. Then $\mathbf{f} \in C^1(M)$ is *structurally stable on* M if here is an $\varepsilon > 0$ such that for all $\mathbf{g} \in C^1(M)$ with

$$\|\mathbf{f} - \mathbf{g}\|_1 < \varepsilon,$$

; is topologically equivalent to $\mathbf{f}$.

Remark 2. In 1962, Peixoto [22] gave a complete characterization of the tructurally stable, C^1 vector fields on any compact, two-dimensional, dif-erentiable manifold M (such as S^2) and he showed that they form a dense, pen subset of $C^1(M)$; cf. Theorem 3 below. However, it was later shown

that on any open, two-dimensional, differentiable manifold E (such as $\mathbf{R}^2$), there is a subset of $C^1(E)$ which is open in the C^1-topology (defined by the C^1-norm) and which consists of structurally unstable vector fields. Nevertheless, in 1982, Kotus, Krych and Nitecki [16] showed how to control the behavior "at infinity" so as to guarantee the structural stability of a vector field on any two-dimensional, differentiable manifold under "strong C^1-perturbation" and they gave a complete characterization of the structurally stable vector fields on $\mathbf{R}^2$.

Example 1. The vector field

$$\mathbf{f}(\mathbf{x}) = \begin{pmatrix} -y \\ x \end{pmatrix}$$

on $\mathbf{R}^2$ is not structurally stable. To see this we let K be any compact subset of $\mathbf{R}^2$ which contains the origin on its interior and show that $\mathbf{f}$ is not structurally stable on K. Let $\|\cdot\|_1$ denote the C^1-norm on K and define the vector field

$$\mathbf{g}(\mathbf{x}) = \begin{pmatrix} -y + \mu x \\ x + \mu y \end{pmatrix}.$$

Then

$$\|\mathbf{f} - \mathbf{g}\|_1 = |\mu|(\max_{\mathbf{x}\in K} |\mathbf{x}| + 1)$$

and if d is the diameter of K, i.e. if

$$d = \max_{\mathbf{x}\in K} |\mathbf{x}|,$$

it follows for all $\varepsilon > 0$ that if we choose $|\mu| = \varepsilon/(d+2)$ then $\|\mathbf{f}-\mathbf{g}\|_1 < \varepsilon$. The phase portraits for the system $\dot{\mathbf{x}} = \mathbf{g}(\mathbf{x})$ are shown in Figure 1. Clearly $\mathbf{f}$ is not topologically equivalent to $\mathbf{g}$; cf. Problem 1. Thus $\mathbf{f}$ is not structurally stable on $\mathbf{R}^2$. The number $\mu = 0$ is called a bifurcation value for the system $\dot{\mathbf{x}} = \mathbf{g}(\mathbf{x})$.

Example 2. The system

$$\begin{aligned} \dot{x} &= -y + x(x^2 + y^2 - 1)^2 \\ \dot{y} &= x + y(x^2 + y^2 - 1)^2 \end{aligned}$$

is structurally unstable on any compact subset $K \subset \mathbf{R}^2$ which contains the unit disk on its interior. This can be seen by considering the system

$$\begin{aligned} \dot{x} &= -y + x[(x^2 + y^2 - 1)^2 - \mu] \\ \dot{y} &= x + y[(x^2 + y^2 - 1)^2 - \mu] \end{aligned}$$

which is ε-close to the above system if $|\mu| = \varepsilon/(d+2)$ where d is the diameter of K. But writing this latter system in polar coordinates yields

$$\begin{aligned} \dot{r} &= r[(r^2 - 1)^2 - \mu] \\ \dot{\theta} &= 1. \end{aligned}$$

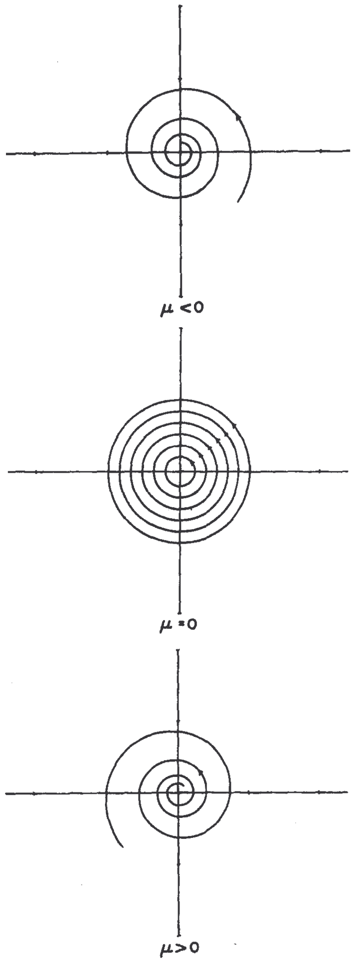


Figure 1. The phase portraits for the system $\dot{\mathbf{x}} = \mathbf{g}(\mathbf{x})$ in Example 1.

Hence, we have the phase portraits shown in Figure 2 below; and the above system with $\mu = 0$ is structurally unstable; cf. Problem 2. The number $\mu = 0$ is called a bifurcation value for the above system and for $\mu = 0$ this system has a limit cycle of multiplicity two represented by $\gamma(t) = (\cos t, \sin t)^T$.

Note that for $\mu = 0$, the origin is a nonhyperbolic critical point for the system in Example 1 and $\gamma(t)$ is a nonhyperbolic limit cycle of the system in Example 2. In general, dynamical systems with nonhyperbolic equilibrium points and/or nonhyperbolic periodic orbits are not structurally stable. This does not mean that dynamical systems with only hyperbolic equilibrium points and periodic orbits are structurally stable; cf., e.g., Theorem 3 below.

Before characterizing structurally stable planar systems, we cite some results on the persistance of hyperbolic equilibrium points and periodic orbits; cf., e.g., [H/S], pp. 305–312.

Theorem 1. *Let $\mathbf{f} \in C^1(E)$ where E is an open subset of $\mathbf{R}^n$ containing a hyperbolic critical point $\mathbf{x}_0$ of (2). Then for any $\varepsilon > 0$ there is a $\delta > 0$ such that for all $\mathbf{g} \in C^1(E)$ with*

$$\|\mathbf{f} - \mathbf{g}\|_1 < \delta$$

there exists a $\mathbf{y}_0 \in N_\varepsilon(\mathbf{x}_0)$ such that $\mathbf{y}_0$ is a hyperbolic critical point of (2′); furthermore, $D\mathbf{f}(\mathbf{x}_0)$ and $D\mathbf{g}(\mathbf{y}_0)$ have the same number of eigenvalues with negative (and positive) real parts.

Theorem 2. *Let $\mathbf{f} \in C^1(E)$ where E is an open subset of $\mathbf{R}^n$ containing a hyperbolic periodic orbit Γ of (2). Then for any $\varepsilon > 0$ there is a $\delta > 0$ such that for all $\mathbf{g} \in C^1(E)$ with*

$$\|\mathbf{f} - \mathbf{g}\|_1 < \delta$$

there exists a hyperbolic periodic orbit Γ' of (2′) contained in an ε-neighborhood of Γ; furthermore, the stable manifolds $W^s(\Gamma)$ and $W^s(\Gamma')$, and the unstable manifolds $W^u(\Gamma)$ and $W^u(\Gamma')$, have the same dimensions.

One other important result for n-dimensional systems is that any linear system

$$\dot{\mathbf{x}} = A\mathbf{x}$$

where the matrix A has no eigenvalue with zero real part is structurally stable in $\mathbf{R}^n$. Besides nonhyperbolic equilibrium points and periodic orbits, there are two other types of behavior that can result in structurally unstable systems on two-dimensional manifolds. We illustrate these two types of behavior with some examples.

NOW THAT WE HAVE A SUPERFICIAL IDEA OF WHAT BIFURCATION THEORY IS ABOUT, LET'S CONSIDER SOME FLUID MECHANICAL EXAMPLES OF HOW THESE IDEAS CAN BE USED TO INTERPRET PHYSICAL BIFURCATIONS OF CAPILLARY SURFACE SHAPES.

EXAMPLE I

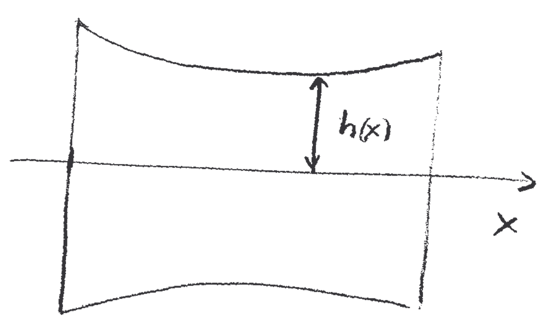
IF YOU MAKE TWO WIRE RINGS OF DIFFERENT SIZE (AS SHOWN), MAKE A SOAP FILM ACROSS THE LARGER RING, AND THEN SLOWLY DRAW THE SOAP FILM OUT W/ THE SMALLER RING AS SHOWN, YOU'LL OBSERVE THE FOLLOWING:

- THE SOAP FILM BETWEEN THE RINGS WILL REMAIN INTACT UNTIL THE SEPARATION BETWEEN THE RINGS IS SLIGHTLY GREATER THAN THE RADIUS, a, OF SAY THE LARGER RING, BUT SMALLER THAN $2a$.
- ONCE SEPN. EXCEEDS THIS VALUE, THE FILM BURSTS AND RETURNS TO ITS ORIGINAL

CONFIGURATION, i.e., AS A CIRCULAR FILM IN THE LARGER RING.

LET'S USE SOME BASIC IDEAS OF BIFURCATION THEORY TO EXPLAIN (OR AT LEAST INTERPRET) THESE OBSERVATIONS.

FIRST OBTAIN (NONLINEAR) DIFF'TL EQN. GOVERNING THE SHAPE OF THE FILM. THIS WILL FOLLOW FROM THE YOUNG-LAPLACE EQN. (i.e., THE NORMAL STRESS BALANCE APPLIED ACROSS THE FILM).



CARRYING OUT A NORMAL STRESS BALANCE ACROSS AN ARBITRARY, INFINITESIMAL ELEMENT OF AREA ON THE FILM LEADS TO THE YOUNG-LAPLACE EQN. (SEE LAST SEMESTER'S NOTES OR DEEN, ANALYSIS OF TRANSPORT PHENOMENA, PP 233-235; 580)

$$P_{IN} - P_{OUT} = \sigma \nabla_H \cdot \underline{n} \qquad (3)$$

WHERE ∇_H IS THE SURFACE DIVERGENCE OPERATOR (WHICH YIELDS DIFFERENTIATIONS IN DIRECTIONS TANGENT TO THE FILM SURFACE). SINCE SURFACE IS PARAMETERIZED IN r-direction, i.e., THE EQN OF THE SURFACE IS

$$r = h(x) \qquad (4)$$

$$\boxed{\nabla_H = \hat{e}_x \frac{\partial}{\partial x} + \frac{\hat{e}_\theta}{r} \frac{\partial}{\partial \theta}} \qquad (5)$$

NOW, LET'S CALCULATE $\hat{n}$ AS FOLLOWS:

LET $G(r,x) = r - h(x) = 0$

$$\Rightarrow \boxed{\hat{n} = \frac{\nabla G}{\sqrt{\nabla G \cdot \nabla G}}} \qquad (6)$$

$$\Rightarrow \nabla G = \left(\hat{e}_x \frac{\partial}{\partial x} + \hat{e}_r \frac{\partial}{\partial r}\right)\left(r - h(x)\right)$$

$$= \hat{e}_r \frac{\partial r}{\partial r} + \hat{e}_x \frac{\partial}{\partial x}(-h(x))$$

$$= \hat{e}_r - h'(x)\,\hat{e}_x$$

(NOTE r & x INDEPENDENT VBLS.)

$$\Rightarrow \boxed{\hat{n} = \frac{\hat{e}_r - h'(x)\,\hat{e}_x}{\sqrt{1 + (h')^2}}} \qquad (7)$$

[NOTE, (7) CAN BE DERIVED GEOMETRICALLY USING SAME APPROACH AS USED LAST SEMESTER.] CHECK (7): WHEN $h = $ CONST, $\hat{n} = \hat{e}_r$, CHECKS WHEN $h'(x) > 0$ (i.e., slope of interface positive), THEN THE X-COMP. OF $\hat{n}$ SHOULD BE DIRECTED IN $-$ X-DIR. $\rightarrow$ CHECKS (AND VICE-VERSA).

NEXT CALCULATE LOCAL CURVATURE OF FILM, $\nabla_H \cdot \hat{n}$:

$$\nabla_H \cdot \hat{n} = \left(\frac{\hat{e}_\theta}{r}\frac{\partial}{\partial\theta} + \hat{e}_x\frac{\partial}{\partial x}\right)\cdot\left[\frac{(\hat{e}_r - h'\hat{e}_x)}{\sqrt{1+(h')^2}}\right] \qquad r=h$$

$$\boxed{\nabla_H \cdot \hat{n} = \boxed{\frac{1}{h(1+(h')^2)^{1/2}} - \frac{\partial}{\partial x}\left(\frac{h'}{\sqrt{1+(h')^2}}\right)}} \qquad (8)$$

NOW NOTE THAT $P_{IN} = P_{OUT} = P_{ATM}$. THUS, EQN (3) BECOMES

$$\boxed{\frac{1}{h(1+(h')^2)^{1/2}} - \frac{d}{dx}\left[\frac{h'}{\sqrt{1+(h')^2}}\right] = 0} \qquad (9)$$

NORMAL STRESS BALANCE ACROSS SOAP FILM

NOTE $\displaystyle \frac{d}{dx}\left[\frac{h'}{\sqrt{1+(h')^2}}\right] = \frac{h''}{\sqrt{1+(h')^2}} - \frac{(h')^2 h''}{(1+(h')^2)^{3/2}}$

$$= \frac{h''}{(1+(h')^2)^{3/2}}$$

$\Rightarrow$ (9) BECOMES $\displaystyle \boxed{\frac{1}{h(1+(h')^2)^{1/2}} - \frac{h''}{(1+(h')^2)^{3/2}} = 0} \qquad (10)$

(10) CAN BE INTEGRATED BY LETTING

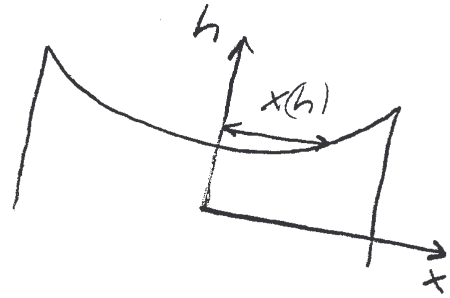
$$f^2 = 1+(h')^2$$

$$\Rightarrow f^{-1} = \left(1+(h')^2\right)^{-1/2}$$

$$= \left(1 + \frac{1}{\left(\frac{dx}{dh}\right)^2}\right)^{-1/2} = \frac{\left(\frac{dx}{dh}\right)}{\sqrt{1+\left(\frac{dx}{dh}\right)^2}}$$

(i.e. use $\frac{dh}{dx} = \frac{1}{\frac{dx}{dh}}$)

NOTE IN ORDER ENSURE A VALID TRANSFORMATION FROM X TO h (i.e., IN ORDER TO BE ABLE TO EXPRESS X AS A FN. OF h,) WE WILL ASSUME THAT THE RADIUS OF THE SECOND RING (USED TO PULL FILM OUT) IS APPROXIMATELY a. THUS, DUE TO SYMMETRY ABOUT X=0, WE CAN EXPRESS X AS A (PROPER) FN. OF h:



IN OTHER WORDS WE CAN DETERMINE SOLN. FOR $X \geq 0$ AND BY SYMMETRY (ABOUT X=0) THUS OBTAIN SOLN. FOR $X \leq 0$.

NOW CALCULATE $\frac{d(f^{-1})}{dh}$:

$$\frac{d(f^{-1})}{dh} = \frac{d}{dh}\left[\left(1+(x')^2\right)^{-1/2} x'\right]$$

$$= \frac{x''}{\sqrt{1+(x')^2}} - \frac{(x')^2 x''}{(1+(x')^2)^{3/2}}$$

$$\Rightarrow \boxed{\frac{d}{dh}(f^{-1}) = \frac{x''}{(1+(x')^2)^{3/2}}} \qquad (11)$$

NOW $$x'' = \frac{d}{dh}(x') = \frac{d}{dh}\left(\frac{1}{h'}\right) = \frac{d}{dx}\left(\frac{1}{h'}\right)\frac{dx}{dh}$$

$$= -\frac{h''}{(h')^2}(x')$$

$$\boxed{x'' = -\frac{h''}{(h')^3}} \qquad (12)$$

USING (12) IN (11) YIELDS

$$\frac{d}{dh}(f^{-1}) = \frac{-h''/(h')^3}{(1+(x')^2)^{3/2}} = \frac{-h''/(h')^3}{((h')^2+1)^{3/2}/(h')^3}$$

$$\boxed{\frac{d}{dh}(f^{-1}) = \frac{-h''}{(1+(h')^2)^{3/2}}} \qquad (13)$$

NOW USING (13) IN (10) LEADS TO

$$\frac{1}{h(1+(h')^2)^{1/2}} + \frac{d}{dh}(f^{-1}) = 0$$

or since $f^{-1} = (1+(h')^2)^{-1/2}$

$$\boxed{\frac{f^{-1}}{h} + \frac{d}{dh}(f^{-1}) = 0} \qquad (14)$$

SEPARATING AND INTEGRATING (14) GIVES

$$\ln(f^{-1}) = -\ln h + C_0$$

$$\Rightarrow \quad f^{-1} = A_0/h \qquad A_0 = \text{CONST}$$

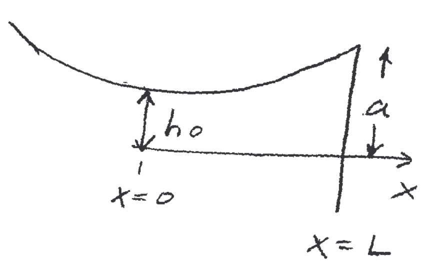
$$\Rightarrow \quad \frac{A_0}{h} = \frac{1}{(1+(h')^2)^{1/2}}$$

$$\Rightarrow \quad \boxed{1+(h')^2 = \frac{h^2}{A_0^2}} \qquad (15)$$

$$\Rightarrow \quad h' = \pm\sqrt{\frac{h^2}{A_0^2} - 1}$$ (USE '+' sign for $x \geq 0$)

$$\int_{h_0}^{a} \frac{A_0\,dh}{(h^2 - A_0^2)^{1/2}} = \int_0^L dx \qquad (16)$$

(NOTE : SEAN BETWEEN RINGS IS DENOTED AS 2L ; ALSO, radius of 2nd ring = a)



Let $\boxed{h = A_0 \cosh u}$ (17)

$\Rightarrow$ $dh = A_0 \sinh u \, du$

(16) $\Rightarrow$

$$\frac{A_0^2}{A_0}\int \frac{\sinh u \, du}{\sqrt{\cosh^2 u - 1}} = A_0 u = x + C_0$$

$\nearrow$ $\sinh u$ $\qquad$ C_0 = integ. const.

$\Rightarrow$ $\boxed{u = \frac{x}{A_0} + b_0}$ $\qquad b_0 = \frac{C_0}{A_0}$ (18)

SO USING (18) IN (17) YIELDS

$$\boxed{h(x) = A_0 \cosh\left[\frac{x}{A_0} + b_0\right]} \qquad (19)$$

NOW USE B.C.'S ON $h(x)$:

$$h(0) = h_0 = A_0 \cosh(b_0) \qquad (20a)$$
$$h(L) = a = A_0 \cosh\left(\frac{L}{A_0} + b_0\right) \qquad (20b)$$

REPLACE FIRST B.C. BY A SYMMETRY CONDITION

$$h'(x=0) = 0 = \sinh(b_0)$$

$$\Rightarrow \boxed{b_0 = 0}$$

$\Rightarrow$ (20a) $\Rightarrow$

$$h_0 = A_0 \cosh(0)$$
$$\Rightarrow \boxed{A_0 = h_0}$$

$$\Rightarrow \boxed{h(x) = h_0 \cosh\left[\frac{x}{h_0}\right]} \quad ** \quad (21)$$

EQ (21) DESCRIBES A <u>CATENOID</u>
(SEE ELEMENTARY CALCULUS BOOK).

FROM (20b) WE NOW OBTAIN

$$a = h_o \cosh\left(\frac{L}{h_o}\right)$$

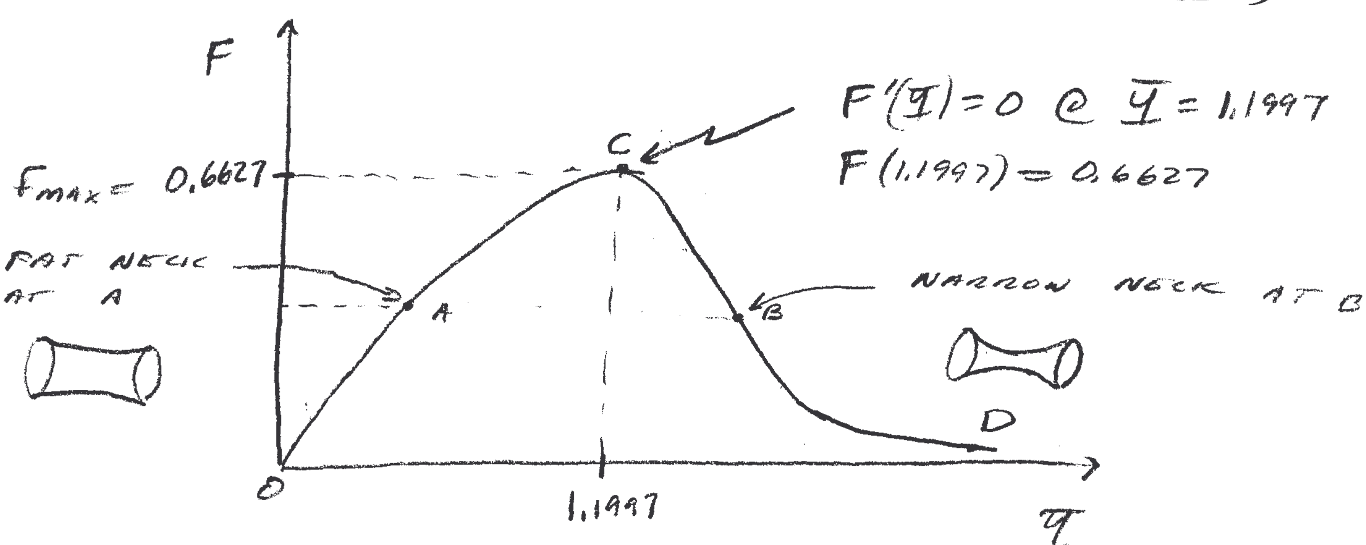
OR

$$aL = h_o L \cosh\left(\frac{L}{h_o}\right)$$

$$\Rightarrow \quad \frac{\left(\frac{L}{h_o}\right)}{\cosh\left(\frac{L}{h_o}\right)} = \frac{L}{a}$$

$$\Rightarrow \quad \boxed{\frac{\Psi}{\cosh \Psi} = F \quad \Bigg| \quad \begin{array}{l} \Psi = \dfrac{L}{h_o} \\ F = \dfrac{L}{a} \end{array}} \qquad (22)$$

($2F$ = DIM'LESS RING SEPARATION)
NOW PLOT SOLN. IN (22) (RESPONSE CURVE)



PLOT SHOWS THAT AT ANY GIVEN (HALF) SEPN. F, THERE ARE 2 SHAPE SOLNS;

SOLNS. ON THE BRANCH OC CORRESPOND TO FILM SURFACES W/ A FAT NECK (SINCE $\bar{g}_A = \frac{L}{h_{o_A}}$ IS RELATIVELY SMALL) WHILE SOLNS ON CD CORRESPOND TO FILMS W/ NARROW NECK.

DISCUSSION CONCERNING PLOT — FOR ANY GIVEN SEPN., 2F, SURFACE AREA A ASSOCIATED W/ THE NARROW NECK WILL BE GREATER THAN THE SURFACE AREA ASS'D. WITH THE FAT NECK. SINCE STATIC CAPILLARY SURFACES ASSUME SHAPES WHICH MINIMIZE SURFACE POTENTIAL ENERGY (ASS'D W/ SURFACE TENSION) AND SINCE THIS ENERGY VARIES W/ SURFACE AREA, WE EXPECT THAT SINCE THE FAT NECK SURFACE HAS LOWER POTENTIAL ENERGY, IT WILL BE THE SOLN ACTUALLY OBSERVED. THIS IS BORE OUT BY OBSERVATION. INDEED, IT IS OBSERVED THAT NARROW NECK SOLNS AS ON CD ARE UNSTABLE.

A SECOND FINDING FROM THE PLOT ON PG. 16 IS THAT STEADY CATENOID SOLNS. [AS GIVEN BY EQ. (21)], ARE NOT POSSIBLE FOR $F > 0.6627$, OR EQUIVALENTLY FOR $\boxed{2L > 1.3254\,a}$. THUS, ONCE THE RINGS ARE PULLED TO A DISTANCE EXCEEDING $\approx \frac{4}{3}$ ring radius, THE FILM BURSTS.

FINALLY, FROM A BIFURCATION THEORY PERSPECTIVE, IT IS WELL KNOWN THAT STABILITY IS LOST AT TURNING POINTS, i.e., AT POINTS WHERE DISTINCT MEMBERS OF A FAMILY OF SOLNS. MEET. IN THIS EXAMPLE, WE HAVE TWO MEMBERS (OR SETS) OF SOLNS. TO (22): THE NARROW NECK AND FAT NECK SOLNS, REPRESENTED RESPECTIVELY BY BRANCHES CD AND DC IN THE PLOT ON PG. 16. THESE TWO SETS OF SOLNS. MEET AT POINT C, THE TURNING PT., WHERE THE STABLE FAT NECK FILM BURSTS.