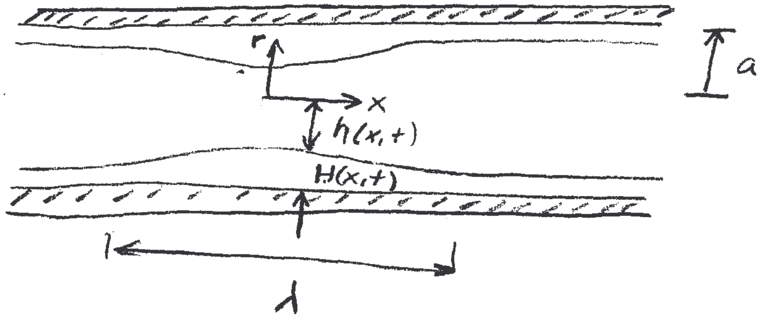
EXAMPLE 3 – CAPILLARY INSTABILITY OF A THIN ANNULAR FILM

THIS PROBLEM CONSIDERS THE LINEAR STABILITY OF THE THIN FILM DISCUSSED IN THE LAST EXAMPLE. WE WILL NOT USE BIFURCATION IDEAS HERE BUT RATHER LINEAR STABILITY IDEAS THAT HAVE ALREADY BEEN DISCUSSED. THE RESULTS GIVE INFORMATION RE. THE EARLY (LINEAR) DYNAMICS OF FILM BREAK-UP INTO COLLARS AND LENSES.

OBJECTIVE: FIND THE COMPLEX GROWTH RATE s FOR AXISYMMETRIC DISTURBANCES TO THE ANNULAR FILM CONSIDERED IN EX. 2.

ONCE WE OBTAIN s, WE'LL SHOW THAT THE FASTEST GROWING UNSTABLE AXISYMMETRIC MODE HAS A WAVELENGTH $\lambda = 2^{3/2}\pi$. THIS PROBLEM WAS FIRST SOLVED BY RAYLEIGH AND IS CALLED RAYLEIGH INSTABILITY.

MODEL



FOCUS ON EARLY-TIME EVENTS WHEN SLOPE, $\frac{\partial H}{\partial x}$, OF FILM INTERFACE IS STILL SMALL. ASSUME THAT FILM THICKNESS, $H = H(x,t)$, IS SMALL (AT ALL x) RELATIVE TO a;

$$\frac{H}{a} << 1 \qquad (1)$$

EXPERIMENTALLY, IT IS OBSERVED THAT UNSTABLE DISTURBANCES (TO THE FILM INTERFACE) HAVE WAVELENGTH λ WHICH IS ON THE ORDER OF THE TUBE'S CIRCUMFERENCE ($2\pi a$). THUS, FROM (1)

$$\frac{H}{\lambda} << 1 \qquad (2)$$

SINCE THE FILM IS THIN AND A STATE OF NO-FLOW INITIALLY EXISTS, WE WILL ALSO ASSUME INERTIAL TERMS

ARE <u>NEGLIGIBLE</u> COMPARED TO THE DOMINANT VISCOUS TERM. THUS, THE MODEL WILL BE BASED ON A <u>LUBRICATION</u> <u>APPROXIMATION</u> (THE SAME ASSUMPTIONS/APPROACH ARE USED TO ANALYZE THIN-FILM FLOW IN BEARINGS).

LET'S ESTIMATE ORDER OF MAGNITUDE OF EACH TERM IN AXIAL MOMENTUM EQN. AND THEN DISCARD SMALL TERMS;

0 ASSUME DISTURBANCE DOES NOT PRODUCE AZIMUTHAL velocity

(3) $$w_{,t} + u w_{,r} + \frac{v w_{,\theta}}{r} + w w_{,z} = -\frac{P_{,x}}{\rho}$$

$$\underbrace{w_{,t}}_{O\left(\frac{W_s}{t_s}\right)} \quad \underbrace{u w_{,r}}_{O\left(\frac{W_s^2 H}{\lambda H}\right)} \quad \underbrace{w w_{,z}}_{O\left(\frac{W_s^2}{\lambda}\right)}$$

$$+ \nu\left[w_{,rr} + \frac{w_{,r}}{r} + \frac{w_{,\theta\theta}}{r^2} + w_{,x}\right]$$

$$O\left(\frac{\nu W_s}{H^2}\right) \quad O\left(\frac{\nu W_s}{Ha}\right) \quad O\left(\frac{W_s \nu}{a^2}\right) \quad O\left(\frac{W_s \nu}{d^2}\right)$$

NOTE WE'VE USED FOLLOWING SCALES

$w \approx W_s$

$x \approx d$

$r \approx a$

$\frac{\partial}{\partial r} \approx \frac{1}{H}$ (← since chgs. in r-direction occur over length scale H)

THE ESTIMATE FOR $uw_{,r}$ FOLLOWS FROM CONTINUITY:

$$u_{,r} + \frac{u}{r} + w_{,x} = 0$$

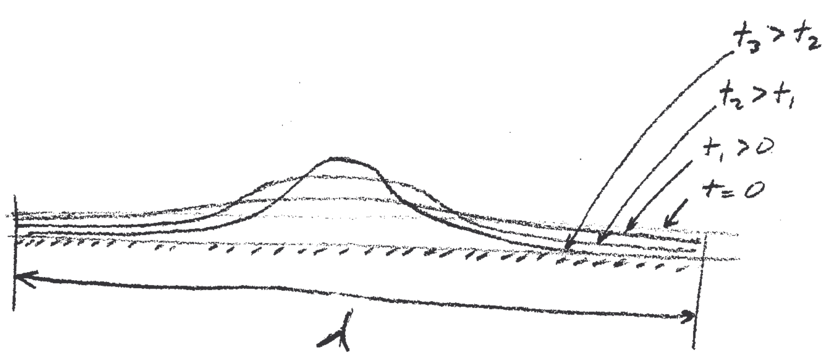
$$O\left(\frac{U_s}{H}\right) \quad O\left(\frac{U_s}{a}\right) \quad O\left(\frac{W_s}{\lambda}\right)$$

SINCE $\frac{U_s}{H} \gg \frac{U_s}{a}$, THEN $u_{,r}$ BALANCES $w_{,z}$. THUS,

$$\frac{U_s}{H} \approx \frac{W_s}{\lambda} \Rightarrow \boxed{U_s \approx W_s \frac{H}{\lambda}} \qquad (4)$$

$$\Rightarrow \quad uw_{,r} \approx U_s \frac{W_s}{H} = \frac{W_s H}{\lambda}\frac{W_s}{H} = \frac{W_s^2}{\lambda}$$

IN ORDER TO ESTIMATE AN APPROPRIATE TIME SCALE, LET'S VISUALIZE WHAT THE FILM LOOKS LIKE AS IT BECOMES UNSTABLE.



FROM THIS PICTURE IT IS CLEAR THAT IF THE INSTABILITY OCCURS OVER CHARACTERISTIC TIME t_s, THE FLUID WITHIN THE FILM WILL MOVE A CHARACTERISTIC DISTANCE λ. THUS, THE CHARACTERISTIC AXIAL VELOCITY w_s ASSOCIATED WITH FLUID MOVEMENT IS RELATED TO λ AND t_s AS FOLLOWS:

$$w_s \approx \frac{\lambda}{t_s} \qquad (5)$$

USING $t_s = \lambda / w_s$ IN EQN. (3), WE SEE THAT $w_{1t} \approx w w_{1z} \approx u w_{1r} = O\left(\frac{w_s^2}{\lambda}\right)$.

NOW LOOKING AT THE VISCOUS TERMS IN (3) IT IS CLEAR THAT THE FIRST TERM IS DOMINANT OVER THE REMAINING 3 TERMS; THUS, WE RETAIN ONLY νw_{1rr}. COMPARING ANY OF THE INERTIAL TERMS W/ νw_{1rr} WE GET

$$\frac{w w_{1z}}{\nu w_{1rr}} = O\left(\frac{w_s^2/\lambda}{\nu w_s/H^2}\right) = O\left(\frac{w_s H}{\nu}\left(\frac{H}{\lambda}\right)\right) \qquad (6)$$

FROM (6), IT IS CLEAR THAT IN ORDER TO NEGLECT THE INERTIAL TERMS COMPARED TO THE DOMINANT VISCOUS TERMS

$$\boxed{\left(\frac{W_s H}{\nu}\right)\left(\frac{H}{a}\right) << 1} \qquad (7)$$

WE'LL ASSUME THIS CONDITION APPLIES; SINCE PRESSURE SCALE IS DETERMINED BY THE DOMINANT TERM FROM AMONG INERTIAL, GRAVITY, AND VISCOUS FORCES AND SINCE VISCOSITY DOMINATES, THEN $P_s = \frac{1 \cdot \mu \cdot W_s}{H^2}$. (IMPORTANTLY, PRESSURE GRADIENT P_{1X} <u>CANNOT</u> BE NEGLECTED IN (3)1.).

THUS, THE X-MOMENTUM EQN SIMPLIFIES TO

$$\boxed{P_{1X} = \mu W_{1rr}} + O\left(\frac{\nu W_s}{Ha}\right) \qquad (8)$$

NOTE THAT SINCE THE LARGEST NEGLECTED TERM IS $\frac{\nu W_{1r}}{r} = O\left(\frac{\nu W_s}{Ha}\right)$, THEN THE IMPLIED ERROR IN (8) IS $O\left(\frac{\nu W_s}{Ha}\right)$, AS SHOWN.

r-momentum (scale analysis):

$$u_{,t} + w u_{,x} + u u_{,r} = -\frac{p_{,r}}{\rho} + \nu\left[u_{,rr} + \frac{u_{,r}}{r} + \cancel{\frac{u_{,\theta\theta}}{r^2}} + u_{,xx}\right] \qquad \frac{\partial}{\partial\theta}=0 \rightarrow 0$$

$$u_{,t}: \; O\left(\frac{w_s H}{\lambda(\lambda/w_s)}\right) \qquad w u_{,x}: \; O\left(w_s \frac{w_s H}{\lambda^2}\right) \qquad u u_{,r}: \; O\left(\frac{w_s^2 H^2}{\lambda^2 H}\right)$$

$$\nu u_{,rr}: \; O\left(\frac{\nu w_s H}{\lambda H^2}\right) \qquad \nu\frac{u_{,r}}{r}: \; O\left(\frac{\nu w_s H}{a H}\right) \qquad \nu u_{,xx}: \; O\left(\nu \frac{w_s H}{\lambda^3}\right)$$

NOTE FROM CONT'Y $u_s = \frac{w_s H}{\lambda}$ (see pg. 38). ALSO, $t_s = \frac{\lambda}{w_s}$, $\frac{\partial}{\partial r} \approx \frac{1}{H}$, $\frac{1}{r} \approx \frac{1}{a}$. FROM ABOVE SCALE ESTIMATES, IT IS CLEAR THAT ALL TERMS ON L.H.S. ARE OF SAME ORDER. IN ADDITION, AS WOULD BE EXPECTED, THE DOMINANT VISCOUS TERM IS $\nu u_{,rr}$. NOW, USING PRESSURE SCALE (PG. 40), $P_s = \frac{\lambda \mu w_s}{H^2}$, LET'S COMPARE $\frac{p_{,r}}{\rho}$ AND ANY ONE OF THE ADVECTION TERMS (ON L.H.S.) AND $\frac{p_{,r}}{\rho}$ AGAINST $\mu u_{,rr}$:

$$\frac{w u_{,x}}{\frac{p_{,r}}{\rho}} \approx \frac{w_s^2 H/\lambda^2}{\frac{\lambda \nu w_s}{H^3}} = \left(\frac{w_s H}{\nu}\right)\left(\frac{H}{\lambda}\right)^3 \ll 1$$

BY CONDITION (7).

$$\frac{\frac{\mu u_{irr}}{p_{ir}}}{\rho} \approx \frac{\frac{\mu W_s H}{\lambda H^2}}{\frac{\lambda \mu W_s}{H^3}} = \left(\frac{H}{\lambda}\right)^2 \ll 1$$ [FROM CONDITION (2)]

THUS, IT IS CLEAR THAT THE r-MOMENTUM EQN. LEADS TO

(relative error = scale of largest neglected term ÷ scale of largest term)

$$(9) \quad \boxed{p_{ir} = 0 + O\left(\left(\frac{H}{\lambda}\right)^2\right)}$$

[NOTE THAT RELATIVE ERROR INTRODUCED BY NEGLECTING ADVECTION TERMS IS $O\left(\frac{W_s H}{\nu}\left(\frac{H}{\lambda}\right)^3\right)$ WHILE RELATIVE ERROR ASSOCIATED W/ NEGLECTING VISCOUS TERMS (DETERMINED BY SCALE OF μu_{irr}) IS $O\left(\left(\frac{H}{\lambda}\right)^2\right)$. SINCE $\frac{W_s H}{\nu}\left(\frac{H}{\lambda}\right)^3 \Big/ \left(\frac{H}{\lambda}\right)^2 = \frac{W_s H}{\nu}\left(\frac{H}{\lambda}\right) \ll 1$, THEN THE LARGEST RELATIVE ERROR IS CLEARLY DUE TO NEGLECTING μu_{irr}. THUS, THE (IMPLIED) ERROR IN EQN (9) IS THIS LARGER ERROR, WHICH IS $O\left(\left(\frac{H}{\lambda}\right)^2\right)$.]

[NOTE 2: THE RELATIVE ERROR IN THE x-MOM. EQN (8) IS $\approx \frac{\nu w_{ir}/r}{\nu w_{irr}} \approx \frac{H}{a} \approx \frac{H}{\lambda} \ll 1$.]

WE HAVE TO SATISFY NORMAL STRESS BOUNDARY CONDITION AT INTERFACE:

$$\sigma \nabla_H \cdot \hat{n} = -P_{IN} + P_{OUT} \qquad (10)$$

LET $P_{OUT} = 0$ (i.e., CONSIDER GAGE PRESSURES w/in FILM). SINCE $P_{IN} = P(x,t)$ (WHERE (9) SHOWS THAT $P \neq fn(r)$), THEN (10) BECOMES

$$\boxed{\sigma \nabla_H \cdot \hat{n} = -P} \qquad (11)$$

NOW CALCULATE CURVATURE $\nabla_H \cdot \hat{n}$:

EQN OF INTERFACE: $r = h(x,t)$

$\Rightarrow \quad G = r - h(x,t)$

$$\Rightarrow \quad \hat{n} = \frac{\nabla G}{|\nabla G|} = \frac{\hat{e}_r - h_{,x}\hat{e}_x}{\sqrt{1+h_{,x}^2}}$$

$$\Rightarrow \quad \nabla_H \cdot \hat{n} = \left(\frac{\hat{e}_\theta}{r}\frac{\partial}{\partial\theta} + \hat{e}_x\frac{\partial}{\partial x}\right)\cdot\left(\frac{\hat{e}_r - h_{,x}\hat{e}_x}{\sqrt{1+h_{,x}^2}}\right)$$

$$= \frac{1}{h(1+h_{,x}^2)^{1/2}} - \frac{h_{,xx}}{(1+h_{,x}^2)^{3/2}}$$

(see last 2 examples for this step)

SINCE WE ARE LIMITING OURSELVES TO A LINEAR ANALYSIS OF FILM DYNAMICS, LET'S LINEARIZE THE CURVATURE TERM. [NOTE, A NONLINEAR ANALYSIS IS POSSIBLE. HOWEVER, BASED ON EXAMPLE 2, SOME NUMERICS WOULD BE NECESSARY — SEE EQN. 2.9 ON PAGE 25 AND NOTE THAT G IS THE DIM'LESS FILM PRESSURE AND THAT $h' \Rightarrow \frac{\partial h}{\partial X}$. WE WOULD HAVE TO SOLVE THE X-MOMENTUM EQN. AND BE SURE THAT $H/\lambda << 1$ IS SATISFIED IN ORDER THAT RELATIVE ERRORS IN X-MOMENTUM EQN. (8) AND Y-MOMOMOMENTUM EQN (9) REMAIN SMALL.] [NOTE 2: WE COULD DEFER LINEARIZATION UNTIL LATER (see pg. 55). THE END RESULT WOULD BE THE SAME.]

FIRST LET $\tilde{H} = a - h(x,t) = \tilde{H}(x,t)$.

THUS, $h_{1x} = -\tilde{H}_{1x}$, $h_{1xx} = -\tilde{H}_{1xx}$ AND

$$\nabla_H \cdot \hat{n} = \underset{(\mathrm{I})}{\frac{1}{(a-\tilde{H})(1+\tilde{H}_{1x}^2)^{1/2}}} + \underset{(\mathrm{II})}{\frac{\tilde{H}_{1xx}}{(1+\tilde{H}_{1x}^2)^{3/2}}} \qquad (11)$$

NOW

$$\frac{1}{a-\tilde{H}} = \frac{1}{a(1-\tilde{H}/a)} = \frac{1}{a}\left(1+\frac{\tilde{H}}{a}\right) \quad \text{(TAYLOR EXPAND FOR SMALL } \tilde{H}/a\text{)}$$

$$\frac{1}{(1+\tilde{H}_{1x}^2)^{1/2}} = \frac{1}{(1+u^2)^{1/2}} = f(u) \quad (u = \tilde{H}_{1x})$$

$$f(u) = f(0) + \overset{-\frac{u}{(1+u^2)^{3/2}}}{f'(0)}\,u + \frac{f''(0)}{2!}u^2 \ldots$$

$$= 1 + (0)u + \frac{1}{2}(-1)u^2 + \ldots$$

$$= 1 + O(\tilde{H}_{1x}^2) = 1 + O\left(\frac{H^2}{\Lambda^2}\right)$$

$\Rightarrow$ TERM (I) IN (11) $\Rightarrow$ $\frac{1}{a}\left(1+\frac{\tilde{H}}{a}\right) + O\left(\frac{H^2}{\Lambda^2}\right)$

LIKEWISE TERM II IN (11) $\Rightarrow$ $\tilde{H}_{1xx} + O\left(\frac{H^2}{\Lambda^2}\right)$

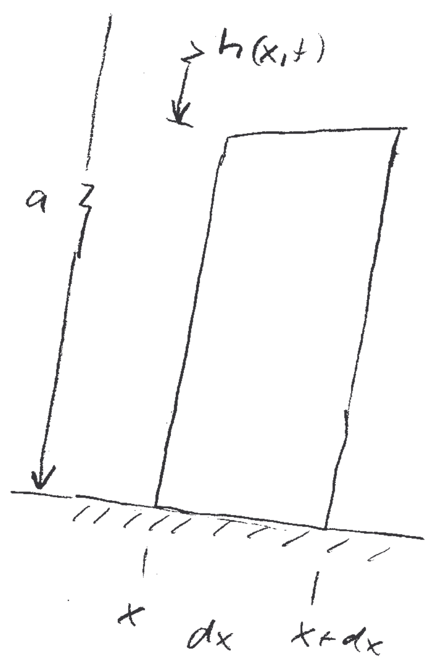
$\Rightarrow$

$$\sigma \nabla_H \cdot \hat{n} = \boxed{\sigma\left[\frac{1}{a}\left(1+\frac{\tilde{H}}{a}\right) + \tilde{H}_{1xx}\right] = -P} \quad (12)$$

WE CAN NOW CALCULATE THE CAPILARY-INDUCED PRESSURE GRADIENT P_{1x} (PRODUCED BY SURFACE TENSION) WHICH DRIVES THE VISCOUS FLOW WITHIN THE FILM [AS DESCRIBED BY THE X-MOMENTUM EQN. (8)]:

$$(12) \Rightarrow -P_{1x} = \boxed{\sigma\left[\frac{\tilde{H}_{1x}}{a^2} + \tilde{H}_{1xxx}\right] = -\mu W_{1rr}} \quad (13)$$

IN ORDER TO OBTAIN AN EVOLUTION EQN. FOR THE INTERFACE, LET'S DO A MASS BALANCE ON A SECTION, dx, OF THE FILM:



IN TIME δt, THE VOLUME OF LIQUID FLOWING INTO THE SLICE AT x IS

$$\int_h^a w\delta t \, 2\pi r \, dr \Big|_x \quad (14a)$$

LIKEWISE THE VOL. OF FLUID EXITING FROM $x+dx$ IS

$$\int_h^a w \Delta t \, 2\pi r \, dr \Big|_{x+dx} \qquad (14b)$$

THE NET GAIN/LOSS OF VOLUME IS THUS

$$\int_h^a w \Delta t \, 2\pi r \, dr \Big|_x - \int_h^a w \Delta t \, 2\pi r \, dr \Big|_{x+dx} =$$

$$-\frac{\partial}{\partial x}\left(\int_h^a w \, 2\pi r \, dr\right)\Big|_x dx \, \Delta t \qquad (14c)$$

THE VOLUME CHANGE THAT OCCURS DUE TO NET FLOW INTO OR OUT OF THE SLICE (OVER TIME Δt) IS GIVEN BY

$$\forall(t+\Delta t) - \forall(t) = \left[\left(\int_h^a 2\pi r \, dr\right)\Big|_{t+\Delta t} - \left(\int_h^a 2\pi r \, dr\right)\Big|_t\right] dx$$

$$= \left[\frac{\partial}{\partial t}\left(\int_h^a 2\pi r \, dr\right)\Delta t\right] dx$$

$$= \frac{\partial}{\partial t}\left[\pi(a^2 - h^2)\right]\Delta t \, dx \qquad (14d)$$

NOW WE CAN SET (14c) EQUAL TO (14d):

$$\boxed{-\frac{\partial}{\partial x}\int_h^a 2wr\,dr = \frac{\partial}{\partial t}\left(a^2 - h^2\right)} \qquad (15)$$

(NOTE, $\pi\, dx \Delta t$ HAS BEEN FACTORED OUT.)

NOW, SOLVE X-MOM. FOR w IN TERMS OF $P_{,x}$, REPLACE $P_{,x}$ W/ MIDDLE TERM IN (13), AND SUBSTITUTE RESULT INTO (15) TO OBTAIN GOVERNING EQN. FOR $\tilde{H}$.

FROM (8) $\quad w_{,rr} = \dfrac{P_{,x}}{\mu}$

$$\Rightarrow w = \frac{P_{,x}}{2\mu} r^2 + C_1 r + C_2$$

(RECALL $P \neq fn(r)$)

B.C.1

$$w(r=a,x) = 0 = \frac{P_{1x}}{2\mu}a^2 + C_1 a + C_2$$

$$C_2 = -\frac{P_{1x}}{2\mu}a^2 - C_1 a$$

B.C.2

$w_{,r}(r=h) = 0$ (ASSUME NO SHEAR STRESS AT FREE-SURFACE)

$$\Rightarrow \quad \frac{P_{1x}}{\mu}h + C_1 = 0 \Rightarrow \boxed{C_1 = \frac{-P_{1x}}{\mu}h}$$

$$\rightarrow \boxed{C_2 = -\frac{P_{1x}}{2\mu}(a^2 - 2ah)}$$

$$\Rightarrow \quad w(r,x) = \frac{-P_{1x}}{2\mu}(-r^2 + 2hr + a^2 - 2ah)$$

$$\boxed{w(r,x) = -\frac{P_{1x}}{2\mu}\left(a^2 - r^2 - 2h(a-r)\right)} \qquad (16)$$

NOW EVALUATE INTEGRAL IN (15) USING (16).

IN ORDER TO SIMPLIFY SUBSEQUENT ALGEBRA, LET'S FIRST SIMPLIFY (16):

$$w = -\frac{P_{1x}}{2\mu}(a-r)(a+r-2h) \qquad (17)$$

LET $y = a - r$, $\tilde{H} = a - h \Rightarrow a + r - 2h = a - h + r - h$
$= \tilde{H} + \tilde{H} - y$

$$\Rightarrow \boxed{w = -\frac{P_{1x}}{2\mu} y(2\tilde{H} - y)} \quad (18)$$

$$\Rightarrow \quad dr = -dy$$

$$\Rightarrow I = \int_h^a 2rw\,dr = -2\int_{\tilde{H}}^0 (a-y)\left(\frac{-P_{1x}}{2\mu}\right) y(2\tilde{H} - y)\,dy$$

$$= -\frac{P_{1x}}{\mu}\int_0^{\tilde{H}} \left(2a\tilde{H}y - (a + 2\tilde{H})y^2 + y^3\right) dy$$

$$= -\frac{P_{1x}}{\mu}\left[a\tilde{H}^3 - (a + 2\tilde{H})\frac{\tilde{H}^3}{3} + \frac{\tilde{H}^4}{4}\right]$$

$$\boxed{I = -\frac{P_{1x}}{\mu}\left(\frac{2}{3}a\tilde{H}^3 - \frac{5}{12}\tilde{H}^4\right)} \quad (19)$$

$\Rightarrow$ USING (13) AND (19), EVOLUTION EQN (15) BECOMES:

$$\boxed{-\frac{\partial}{\partial x}\left[\frac{\sigma\left[\frac{\tilde{H}_{1x}}{a^2} + \tilde{H}_{1xxx}\right]}{\mu}\left(\frac{2}{3}a\tilde{H}^3 - \frac{5}{12}\tilde{H}^4\right)\right] = 2(a - \tilde{H})\frac{\partial \tilde{H}}{\partial t}} \quad (20)$$

THIS EQN., WHICH GOVERNS THE TIME AND POSITION-DEPENDENT EVOLUTION OF THE FILM INTERFACE, HAS <u>TWO</u> <u>STEADY</u> <u>SOLNS</u>.

SOLN. <u>1</u> : (SET $\frac{\partial \tilde{H}}{\partial t} = 0$)

$\boxed{\tilde{H} = \text{CONST} = \tilde{H}_0}$. (NOTE $\frac{\tilde{H}_{0,x}}{a^2} + \tilde{H}_{0,xxx} = 0$)

PHYSICALLY, THIS CORRESPONDS TO AN <u>ANNULAR FILM</u>. WE WILL SHOW BELOW THAT THE ANNULAR FILM SOLN IS <u>UNSTABLE</u>, CONSISTENT WITH EXPERIMENTAL OBSERVATIONS.

STEADY SOLN, 2 :

$$\boxed{\frac{\tilde{H}}{a^2} + \tilde{H}_{,xx} = \text{CONSTANT} = C_0 .} \qquad (21)$$

(IN THIS CASE, AGAIN, $\frac{\tilde{H}_{,x}}{a^2} + \tilde{H}_{,xxx} = \left(\frac{\tilde{H}}{a^2} + \tilde{H}_{,xx}\right)_{,x} = 0$)

WE CAN EASILY INTEGRATE (21) TO OBTAIN THE CORRESPONDING CAPILLARY SURFACE SHAPE ;

$$\left(D^2 + \frac{1}{a^2}\right)\tilde{H} = C_0 \qquad \left(D^2 = \frac{d^2}{dx^2}\right)$$

$$\Rightarrow \tilde{H} = A_0 \sin\left(\frac{x}{a}\right) + B_0 \cos\left(\frac{x}{a}\right) + C_0 a^2$$

BEFORE FIGURING OUT A_0 & B_0, LET'S LOOK AT (21) AND RECOGNIZE THAT THIS IS NONE OTHER THAN A <u>LINEARIZED</u> VERSION OF THE NORMAL STRESS B.C. AT THE FILM INTERFACE. [THIS CAN BE SHOWN STARTING W/ $\Delta P = \sigma\left[\frac{h''}{(1+(h')^2)^{3/2}} + \frac{1}{h(1+(h')^2)^{1/2}}\right]$ THEN BY SUBSTITUTING $h = a - \tilde{H}$, AND FINALLY BY LINEARIZING (AS WAS DONE REFERING TO OBTAIN EQN. (12) ON PG. (45))].

THE LINEARIZED NORMAL STRESS B.C. GIVEN BY EQN. (12) (PG. 45), WE SEE THAT

$$c_0 = \frac{P}{\sigma} - \frac{1}{a} = \frac{1}{\sigma}\underbrace{\left(P - \frac{\sigma}{a}\right)}$$

PRESSURE difference BETWEEN ACTUAL AND PRESSURE IN A CAPILLARY FILM $\left(\equiv \frac{\sigma}{a}\right)$

IF WE TAKE A HINT FROM COLLAR AND LENS SOLNS FROM EX. 2 AND ASSUME SYMMETRY ABOUT $x=0$ AND THAT $\tilde{H}_{1x}=0$ AT SOME VALUE OF $x=L$, THEN

SYMMETRY $\Rightarrow$ $\tilde{H}_{1x}=0 \quad @x=0 \Rightarrow \quad \underline{A_0=0}$

$\tilde{H}_{1x}=0 \quad @x=L \Rightarrow -\frac{B_0}{a}\sin\left(\frac{L}{a}\right)=0$

$\Rightarrow \sin\left(\frac{L}{a}\right)=0$

$\Rightarrow L = n\pi a \quad , n=0,1,2 \ldots$

HOWEVER, WE REJECT $n=0$ CASE SINCE THAT IMPLIES A ZERO-LENGTH CAPILLARY FILM. IT TURNS OUT THAT $n=1$ IS THE ONLY PHYSICALLY REALISTIC SOLN. TO PROVE THIS WE WOULD

PRESUMABLY HAVE TO CALCULATE CAPILLARY SURFACE AREAS ASSOCIATED W/ EACH CHOICE OF n AND SHOW THAT MINIMUM AREA (FOR A GIVEN INITIAL VOLUME) CORRESPONDS TO $n=1$.
WE WON'T ATTEMPT THIS HERE.

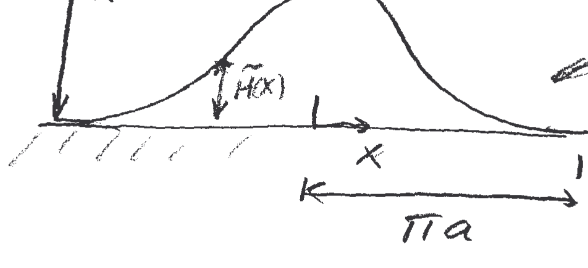
THUS, $\tilde{H}_{1x} = 0$ AT $x = L = \pi a \Rightarrow$ THE OVERALL LENGTH OF THE STEADY CAPILLARY SHAPE IS EQUAL TO THE CIRCUMFERENCE OF THE TUBE.

$$\Rightarrow \quad \tilde{H}(x) = B_0 \cos\left(\frac{x}{a}\right) + \left(\frac{P}{\sigma} - \frac{1}{a}\right)a^2$$

CALLING $\tilde{H}_{MAX} = \tilde{H}_0$ (WHICH IS LOCATED AT $x=0$), WE CAN SOLVE FOR B_0 IN TERMS OF $\tilde{H}_0$:

$$B_0 = \tilde{H}_0 - \left(\frac{P}{\sigma} - \frac{1}{a}\right)a^2$$

$$\Rightarrow \quad \boxed{\tilde{H}(x) = \tilde{H}_0 \cos\left(\frac{x}{a}\right) + \left(\frac{P}{\sigma} - \frac{1}{a}\right)a^2\left(1 - \cos\frac{x}{a}\right)} \quad (22)$$



THIS IS A <u>COLLAR</u>.

STABILITY OF ANNULAR FILM $\tilde{H} = \tilde{H}_0$

TO INVESTIGATE STABILITY, LET'S FOLLOW SAME PROCEDURE USED PREVIOUSLY: 1) LINEARIZE GOV. EQN (20), 2) ASSUME A NORMAL MODE SOLN, 3) OBTAIN COMPLEX GROWTH RATE

LINEARIZE (20) ABOUT $\tilde{H} = \tilde{H}_0$. IN OTHER WORDS LET PERTURBED INTERFACE BE GIVEN BY

$$\tilde{H} = \tilde{H}_0 + H' \qquad (23)$$

PLUG INTO (20), AND NEGLECT QUADRATIC & HIGHER ORDER TERMS IN H':

$$\frac{\sigma}{24a\mu}\frac{\partial}{\partial x}\left[\left(\frac{H'_{x}}{a^2} + H'_{xxx}\right)\left[8a\tilde{H}_0^3 - 5\tilde{H}_0^4\right]\right]$$

$$+ \left(1 - \frac{\tilde{H}_0}{a}\right)\frac{\partial H'}{\partial t} = 0 \qquad (24)$$

WE CAN FURTHER SIMPLIFY THIS EQN BY NOTING THAT THE IMPLIED ERROR IN OUR MODEL IS $O\left(\frac{H}{a}\right)$; THUS

$$1 - \frac{\tilde{H}_0}{a} = 1 + O\left(\frac{H}{a}\right)$$

$$8a\tilde{H}_0^3 - 5\tilde{H}_0^4 = a\tilde{H}_0^3\left(8 - 5\frac{\tilde{H}_0}{a}\right) \doteq 8a\tilde{H}_0^3 + O\left(\frac{H}{a}\right)$$

THUS (24) SIMPLIFIES TO

(25) $$\boxed{\frac{\sigma \tilde{H}_0^3}{3\mu}\left(\frac{H'_{,xx}}{a^2} + H'_{,xxxx}\right) + \frac{\partial H'}{\partial t} = 0} + O\left(\frac{H}{a}\right)$$

TRY $\boxed{H' = f(x)e^{st}}$ NORMAL MODE

(25) $\Rightarrow$

$$f'''' + \frac{f''}{a^2} + sC_2 f = 0 \qquad (26)$$

WHERE $C_2 = \dfrac{3\mu}{\sigma \tilde{H}_0^3}$

ASSUME $f = \sin(kx)$ or $\cos(kx) \rightarrow$ (NOTE, ANY DISTURB. CAN BE DECOMPOSED INTO FOURIER SIN OR COS EXPANSIONS — VALID NORMAL MODES)

(26) $\Rightarrow$ $K^4 - \dfrac{K^2}{a^2} + SC_2 = 0$

$$\Rightarrow \boxed{S = \left(\frac{K^2}{a^2} - K^4\right)\left(\frac{\sigma \tilde{H}_0^3}{3\mu}\right)} \; * \; (27)$$

*

NOTES : 1) $Im(S) = 0$ FOR ALL WAVE NUMBERS, THUS OSCILLATORY, STABLE SOLNS NOT POSSIBLE — MAKES PHYSICAL SENSE SINCE THE FILM FLOW IS DOMINATED BY VISCOSITY; VISCOSITY IS DISSIPATIVE, i.e., WILL NOT ALLOW SELF-SUSTAINED WAVES SUCH AS THOSE ON INVISCID WAVY SURFACES.

2) $Re(s) \geq 0$ FOR $\frac{K^2}{a^2} \geq K^4$, i.e., FOR $K \leq \frac{1}{a}$. THUS, ANY DISTURBANCE HAVING WAVELENGTH $\lambda \geq 2\pi a$ WILL BE UNSTABLE.

3) $Re(s) < 0$ FOR $\frac{K^2}{a^2} < K^4$, i.e. FOR $K > \frac{1}{a}$. THUS, ANY DISTURBANCE W/ WAVELENGTH $\lambda < 2\pi a$ WILL BE STABLE.

4) THE FASTEST GROWING UNSTABLE DISTURBANCE CAN BE FOUND BY FINDING S_{max} (FOR GIVEN $\tilde{H}_0, a, \sigma, \mu$):

$$\frac{ds}{dK} = \frac{\sigma \tilde{H}_0^3}{3\mu}\left(\frac{2K}{a^2} - 4K^3\right) = 0$$

$$\Rightarrow \quad K^2 = \frac{1}{2a^2}$$

$$\Rightarrow \quad K = \frac{1}{\sqrt{2}a}$$

THUS, FASTEST GROWING WAVELENGTH WILL BE $\boxed{\lambda = 2\pi\sqrt{2}a}$.

5) NOTE, WE CARRIED OUT STABILITY ANALYSIS IN A SOMEWHAT 'BACKWARDS' FASHION: i) WE FIRST CALCULATED EVOLUTION EQN. (20); ii) WE CARRIED OUT LINEARIZATION IN TWO STEPS (LINEARIZED NORMAL STRESS B.C. ON PG. 45 AND THEN LINEARIZED REST OF EVOLUTION EQN (20) ON PG. 55); iii) RATHER THAN LINEARIZE NAVIER STOKES EQNS ASSOCIATED W/ PERTURBED FLOW, WE USED SCALE ANALYSIS TO SIMPLIFY THESE EQNS. IF WE ATTEMPTED THE 1st APPROACH, WE WOULD PICK UP A $\frac{\partial w'}{\partial t}$ TERM. IN ORDER [illegible] WITH H [illegible]

SOME NOTES ON PHYSICS OF SOLN.

1) HAVING FOUND THAT THE FASTEST GROWING WAVELENGTH IS $2^{3/2}\pi a \approx 10a$, WE CAN NOW SEE EXACTLY HOW CAPILLARITY DESTABILIZES AND DRIVES THE ANNULAR FILM INTO LENSES OR COLLARS. REFERING TO THE LINEARIZED NORMAL STRESS B.C. (LINEARIZED YOUNG-LAPLACE EQN.) (12) on pg. 45,

$-P = \sigma\left[\frac{1}{a} + \frac{\tilde{H}}{a^2} + \tilde{H}_{1xx}\right]$, WE CAN NOW

REFINE OUR INITIAL ESTIMATE OF THE X-LENGTH SCALE AS $X_s = \lambda \approx 10a$. THUS, SCALING TERMS IN (12) WE GET

$$-P = -P_{IN} = \sigma\left[\frac{1}{a} + \frac{\tilde{H}}{a^2} + \tilde{H}_{1xx}\right]$$

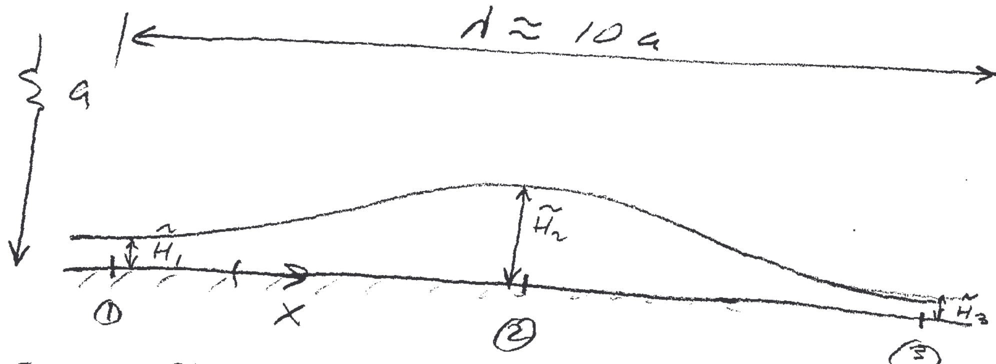
$$= O\left(\frac{\sigma}{a}\right) \quad O\left(\frac{\tilde{H}_0}{a^2}\right) \quad O\left(\frac{\tilde{H}_0}{10^2 a^2}\right)$$

IT IS CLEAR THAT THE SECOND TERM $\left(\frac{\tilde{H}}{a^2}\right)$ IS SIGNIFICANTLY LARGER THAN $\tilde{H}_{1xx}$.

THE FIRST TERM IS A CONSTANT PRESSURE DUE TO SURFACE CURVATURE IN THE AZIMUTHAL DIRECTION [i.e., THE

THE PRESSURE DIFFERENCE, $P_{OUT} - P_{IN}$, ACROSS A THIN, ANNULAR FILM OF THICKNESS $\tilde{H}_0$ IS $P_{OUT} - P_{IN} = \dfrac{\sigma}{(a - \tilde{H}_0)} = \dfrac{\sigma}{a} + O\left(\dfrac{H_0}{a}\right)$.

THIS CONSTANT 'BACKGROUND' PRESSURE PLAYS NO ROLE IN THE FLUID DYNAMICS (SINCE $W \propto \dfrac{\partial P}{\partial x}$). THUS, CAPILLARY-INDUCED FLOW IS DETERMINED ONLY BY THE LAST TWO TERMS IN (12). HOWEVER, SINCE $\dfrac{\tilde{H}}{a^2} \gg \tilde{H}_{xx}$, AT LEAST FOR THE FASTEST GROWING WAVELENGTH, THEN WE CAN DRAW THE FOLLOWING PICTURE AND CLEARLY SEE HOW SURFACE TENSION DRIVES THE FILM FLOW.



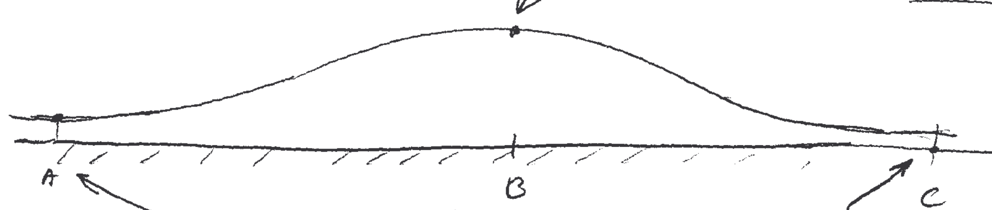
$P_1 \approx -\dfrac{\sigma \tilde{H}_1}{a^2}$ $\qquad P_2 \approx -\dfrac{\sigma \tilde{H}_2}{a^2}$ $\qquad P_3 = P_1 \approx -\dfrac{\tilde{H}_3 \sigma}{a^2}$

SINCE $\tilde{H}_1 < \tilde{H}_2$ THEN P_1 IS LESS NEGATIVE THAN P_2; THUS $P_1 > P_2$. HENCE, THE FLUID MUST

FLOW FROM PT. ① TOWARD PT. ②. LIKEWISE, SINCE $P_3 > P_2$, FLUID MUST FLOW FROM ③ TO ②. THE OBVIOUS RESULT OF THIS INWARD FLOW TOWARD THE CENTER (@ ②) IS INCREASING THINNING AT THE ENDS (PTS ① & ③) AND THICKENING AT THE CENTER. (NOTE, THE $\frac{\tilde{H}}{a^2}$ TERM IS DUE TO CURVATURE IN AZIMUTHAL DIRECTION -- AZIMUTHAL CURVATURE THUS DRIVES THE INWARD FLOW.)

2) JUST AS <u>AZIMUTHAL CURVATURE</u> DRIVES <u>AXIAL</u> INWARD FLOW, <u>AXIAL CURVATURE</u>, REPRESENTED BY $\tilde{H}_{,xx}$ IN (12), FINALLY ACTS TO STOP, OR AT LEAST SLOW INWARD FLOW. THIS CAN BE SEEN FROM THE FOLLOWING PICTURE:

LOCAL MAX IN $\tilde{H} \Rightarrow$ $\underline{\underline{\tilde{H}_{,xx} < 0}}$



LOCAL MINIMA AT Ⓐ & Ⓒ $\Rightarrow$ $\underline{\underline{\tilde{H}_{,xx} > 0}}$

AS MORE FLUID FLOWS INTO CENTER AT (B), THE AXIAL CURVATURE $\tilde{H}_{1xx}$ AT (B) BECOMES LARGER AND MORE NEGATIVE WHILE $\tilde{H}_{1xx}$ AT (A) AND (C) DECREASES (DUE TO FILM THINNING) WHILE REMAINING POSITIVE.

THUS,

$$P_B = -\frac{\sigma}{a} + \underbrace{\left(-\frac{\tilde{H}_B}{a^2}\right)}_{\text{BECOMES INCREASINGLY } -\text{ive AS } \tilde{H}_B \uparrow} + \underbrace{\left(-\tilde{H}_{B_{1xx}}\right)}_{\text{BECOMES INCREASINGLY } +\text{ive AS } \tilde{H}_{B_{1xx}} \text{ BECOMES MORE NEGATIVE}}$$

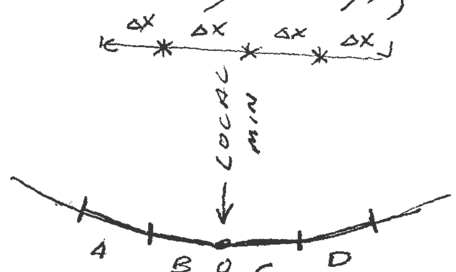
$$P_A = -\frac{\sigma}{a} + \left(\frac{-H_A}{a^2}\right) + \underbrace{\left(-\tilde{H}_{A_{1xx}}\right)}_{\text{BECOMES LESS } -\text{IVE AS FILM THINS AT A}}$$

SO, ONCE $-\sigma(\tilde{H}_{B_{1xx}} + \tilde{H}_B)$ BECOMES LARGER OR EQUALS $-\sigma(H_{A_{1xx}} + \tilde{H}_A)$, INWARD FLOW CEASES. WE CAN SURMISE THAT THE FINAL, STABLE SURFACE SHAPE REPRESENTS A BALANCE BETWEEN THE 'PINCHING' ACTION OF AZIMUTHAL CURVATURE (DRIVING INWARD FLOW) AND THE RETARDING EFFECT OF AXIAL CURVATURE.

(3) WE CAN SHOW THAT AXIAL FLUID FLOW GOES TO ZERO (STAGNATES) ALONG VERTICAL LINES BENEATH LOCAL <u>MAXIMA</u> AND <u>MINIMA</u> IN THE FILM SURFACE. <u>PROOF:</u> SINCE w IS PROPORTIONAL TO P_{1x} (see eqn (16) pg. 49) AND SINCE

$$P_{1x} = -\sigma\left[\frac{\tilde{H}_{1x}}{a^2} + \tilde{H}_{1xxx}\right] \quad \text{(see eqn. 12 pg. 45)},$$

THEN SINCE $\tilde{H}_{1x} = \tilde{H}_{1xxx} = 0$ AT LOCAL MAX. AND MINS. IN SURFACE, $w = 0$ AT MAX/MINS. [IN ORDER TO SHOW THAT $\tilde{H}_{1xxx} = 0$ AT MAX/MINS, LET'S DRAW AN EXPANDED PICTURE OF SURFACE NEAR, e.g., A MINIMUM:



$\tilde{H}_{1xA}$ $\quad\tilde{H}_{1xB}$ $\quad\tilde{H}_{1xC}$ $\quad\tilde{H}_{1xD}$ ← slopes of segments A, B, C, & D.

(BY SYMMETRY ABOUT MIN)

$$\tilde{H}_{1xx_{AB}} \approx \frac{\tilde{H}_{1xB} - \tilde{H}_{1xA}}{\Delta x} \qquad \tilde{H}_{1xx_{CD}} \approx \frac{\tilde{H}_{1xD} - \tilde{H}_{1xC}}{\Delta x} = \frac{-\tilde{H}_{1xA} - (-H_{1xB})}{\Delta x}$$

$$\therefore \boxed{\tilde{H}_{1xx_{CD}} = \tilde{H}_{1xx_{AB}}} \Rightarrow \boxed{\tilde{H}_{1xxx}\big|_0 \approx (\tilde{H}_{xx_{CD}} - \tilde{H}_{1xx_{AB}})/\Delta x = 0}$$

END OF ...

THUS, WE HAVE THE FOLLOWING PICTURE OF THE FLOW FIELD W/IN THE FILM AS IT DESTABILIZES: